

Sobolev L_p^2 -functions on closed subsets of \mathbf{R}^2

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Abstract

For each $p > 2$ we give intrinsic characterizations of the restriction of the homogeneous Sobolev space $L_p^2(\mathbf{R}^2)$ to an arbitrary finite subset E of \mathbf{R}^2 . The trace criterion is expressed in terms of certain weighted oscillations of the second order with respect to a measure generated by the Menger curvature of triangles with vertices in E .

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1. Introduction.

Let $p \in [1, \infty]$ and let m be a positive integer. By $L_p^m(\mathbf{R}^n)$ we denote the homogeneous Sobolev space consisting of all (equivalence classes of) real valued functions $F \in L_{p,loc}(\mathbf{R}^n)$ whose distributional partial derivatives of order m belong to the space $L_p(\mathbf{R}^n)$. The space $L_p^m(\mathbf{R}^n)$ is equipped with the seminorm

$$\|F\|_{L_p^m(\mathbf{R}^n)} := \|\nabla^m F\|_{L_p(\mathbf{R}^n)}$$

where

$$\nabla^m F(x) := \left(\sum_{|\alpha|=m} (D^\alpha F(x))^2 \right)^{\frac{1}{2}}, \quad x \in \mathbf{R}^n.$$

When $p > n$, it follows from the Sobolev embedding theorem that every function $F \in L_p^m(\mathbf{R}^n)$ coincides almost everywhere with a C^{m-1} -function. This fact enables us *to identify each element $F \in L_p^m(\mathbf{R}^n)$, $p > n$, with its unique C^{m-1} -representative*. In particular, this identification implies that *F has a well defined restriction to any given subset of \mathbf{R}^n* . It also enables us to identify the Sobolev space $L_\infty^m(\mathbf{R}^n)$ with the space $C^{m-1,1}(\mathbf{R}^n)$ of C^{m-1} -functions whose partial derivatives of order $m-1$ are Lipschitz continuous on \mathbf{R}^n . The space $C^{m-1,1}(\mathbf{R}^n)$ is equipped with the seminorm

$$\|F\|_{C^{m-1,1}(\mathbf{R}^n)} = \sum_{|\alpha|=m-1} \|D^\alpha F\|_{\text{Lip}(\mathbf{R}^n)}.$$

In this paper we study the following

Problem 1.1 Given a finite set $E \subset \mathbf{R}^2$ and a function $f : E \rightarrow \mathbf{R}$, we consider the $L_p^2(\mathbf{R}^2)$ -norms of all C^1 -functions $F : \mathbf{R}^2 \rightarrow \mathbf{R}$ which coincide with f on E . How small can they be?

We denote the infimum of all these norms by $\|f\|_{L_p^2(\mathbf{R}^2)|_E}$; thus

$$\|f\|_{L_p^2(\mathbf{R}^2)|_E} := \inf\{\|F\|_{L_p^2(\mathbf{R}^2)} : F \in L_p^2(\mathbf{R}^2) \cap C^1(\mathbf{R}^2), F|_E = f\}.$$

As is customary, we refer to $\|f\|_{L_p^2(\mathbf{R}^2)|_E}$ as the *trace norm of the function f* (in $L_p^2(\mathbf{R}^2)$). This quantity provides the standard quotient space seminorm in *the trace space $L_p^2(\mathbf{R}^2)|_E$* of all restrictions of $L_p^2(\mathbf{R}^2)$ -functions to E , i.e., in the space

$$L_p^2(\mathbf{R}^2)|_E := \{f : E \rightarrow \mathbf{R} : \text{there exists } F \in L_p^2(\mathbf{R}^2) \cap C^1(\mathbf{R}^2) \text{ such that } F|_E = f\}.$$

Problem 1.1 is a variant of a classical extension problem posed by H. Whitney in 1934 in his pioneering papers [44, 45], namely: *How can one tell whether a given function f defined on an arbitrary subset $E \subset \mathbf{R}^n$ extends to a C^m -function on all of \mathbf{R}^n ?* Over the years since 1934 this problem, often called the Whitney Extension Problem, has attracted a lot of attention, and there is an extensive literature devoted to different aspects of this problem and its analogues for various spaces of smooth functions. Among the multitude of results we mention the papers by G. Glaeser [22], Y. Brudnyi, P. Shvartsman [5, 6, 7, 33, 34, 38, 36], E. Bierstone, P. Milman and W. Pawlucki [3, 4], and N. Zobin [46, 47]. In the last decade C. Fefferman [9, 14, 13, 10, 16, 19] made several important breakthroughs in this area and developed, partially in collaboration with B. Klartag, a series of new directions of investigation related to computational and algorithmic aspects of the Whitney Extension Problem, see [15, 17, 18, 19]. We refer the reader to all of the above-mentioned papers, and references therein, for numerous results and techniques concerning this topic.

All of these papers deal with functions whose partial derivatives are bounded and continuous or in L_∞ . It is natural to ask analogous questions when L_∞ is replaced by L_p , i.e., to study the extension and restriction properties of functions whose partial derivatives belong to the space L_p with $p < \infty$. In the papers [39, 41] we have considered such questions and presented several constructive descriptions of the trace space $L_p^1(\mathbf{R}^n)|_E$ for an arbitrary subset $E \subset \mathbf{R}^n$ provided $p > n$. (See Remark 1.8 for more details.)

In this paper for $n = 2$ we extend these results from the case of first order derivatives to second order derivatives. More explicitly, we solve Problem 1.1 by presenting a constructive formula for calculation of the order of magnitude of the trace norm $\|f\|_{L_p^2(\mathbf{R}^2)|_E}$. This formula is expressed only in terms of the values of the function f on E and certain geometric characteristics of the set E .

We also prove the existence of a *continuous linear extension operator* from $L_p^2(\mathbf{R}^2)|_E$ to $L_p^2(\mathbf{R}^2)$. Note that the first results related to the existence of continuous linear extension operators acting on traces of Sobolev space to *arbitrary* closed subsets were recently obtained by A. Israel [24] and by C. Fefferman, A. Israel and G. K. Luli [20]. (These results are discussed after Theorem 1.12.)

Before we formulate the main result of the paper we need to define several notions and fix some notation:

Throughout this paper, the word “square” will mean a closed square in \mathbf{R}^2 whose sides are parallel to the coordinate axes. For each square Q we let c_Q denote its center. Given $\lambda > 0$ we let λQ denote the dilation of Q with respect to its center by a factor of λ . The Lebesgue measure of a measurable set $A \subset \mathbf{R}^2$ will be denoted by $|A|$. By $\#A$ we denote the number of elements of a finite set A . Let \mathcal{A} be a family of sets in \mathbf{R}^n . By $M(\mathcal{A})$ we denote its covering multiplicity, i.e., the minimal positive integer M such that every point $x \in \mathbf{R}^n$ is covered by at most M sets from \mathcal{A} .

By $\|\cdot\|$ and $\|\cdot\|_2$ we denote, respectively, the uniform and the Euclidean norms in \mathbf{R}^2 . For each pair of points z_1 and z_2 in \mathbf{R}^2 we let (z_1, z_2) denote the open line segment joining them.

The word “triangle” will always mean a subset $\Delta = \{z_1, z_2, z_3\} \subset \mathbf{R}^2$ consisting of three non-collinear points. By $\text{Triangle}(E)$ we denote the family of all triangles with vertices in E .

We let R_Δ denote the radius of the circle passing through the points z_1, z_2, z_3 . The reciprocal of R_Δ , the quantity

$$(1.1) \quad \mathbf{c}_\Delta := \frac{1}{R_\Delta},$$

is called *the Menger curvature* of the “triangle” Δ (as discussed in detail, e.g. in [31]).

Given a function $f : E \rightarrow \mathbf{R}$ and a triangle $\Delta = \{z_1, z_2, z_3\} \subset E$ we let $P_\Delta[f]$ denote the affine polynomial which interpolates f on Δ . Thus

$$P_\Delta[f] \in \mathcal{P}_1 \quad \text{and} \quad P_\Delta[f](z_i) = f(z_i), \quad i = 1, 2, 3.$$

Here \mathcal{P}_1 denotes the space of polynomials on \mathbf{R}^2 of degree at most one.

Here now is the main result of our paper:

Theorem 1.2 *Let $2 < p < \infty$. Let E be a finite subset of \mathbf{R}^2 and let f be a function defined on E . Then*

$$(1.2) \quad \|f\|_{L_p^2(\mathbf{R}^2)|_E} \sim \inf \lambda^{\frac{1}{p}}$$

where the infimum is taken over all positive constants λ which satisfy all of the following conditions for a certain absolute positive constant γ :

(i). For every finite family $\{Q_i : i = 1, \dots, m\}$ of pairwise disjoint squares and every choice of collinear points $z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \in E$ such that $z_2^{(i)} \in (z_1^{(i)}, z_3^{(i)})$ and

$$z_j^{(i)} \in \gamma Q_i, \quad i = 1, \dots, m, \quad j = 1, 2, 3,$$

the following inequality

$$(1.3) \quad \sum_{i=1}^m \left| \frac{f(z_1^{(i)}) - f(z_2^{(i)})}{\|z_1^{(i)} - z_2^{(i)}\|_2} - \frac{f(z_2^{(i)}) - f(z_3^{(i)})}{\|z_2^{(i)} - z_3^{(i)}\|_2} \right|^p (\text{diam } Q_i)^{2-p} \leq \lambda$$

holds.

(ii). Let \mathcal{Q} and \mathcal{K} be arbitrary finite families of pairwise disjoint squares. Suppose that to each square $K \in \mathcal{K}$ we have arbitrarily assigned a triangle $\Delta(K)$ in E such that

$$(1.4) \quad \Delta(K) \subset \gamma K \quad \text{and} \quad \text{diam } K \leq \gamma \text{diam } \Delta(K).$$

Suppose that to each square $Q \in \mathcal{Q}$ we have arbitrarily assigned two squares $Q', Q'' \in \mathcal{Q}$ such that $Q' \cup Q'' \subset \gamma Q$.

Let

$$(1.5) \quad \sigma_p(Q; \mathcal{K}) := \sum \left\{ \mathbf{c}_{\Delta(K)}^p |K| : K \in \mathcal{K}, c_K \in Q \right\}, \quad Q \in \mathcal{Q},$$

and

$$S_p(f : Q', Q''; \mathcal{K}) := \sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in Q'}} \sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in Q''}} \|\nabla P_{\Delta(K')}[f] - \nabla P_{\Delta(K'')}[f]\|^p \mathbf{c}_{\Delta(K')}^p |K'| \mathbf{c}_{\Delta(K'')}^p |K''|.$$

Then the following inequality

$$\sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} S_p(f : Q', Q''; \mathcal{K})}{\{(\text{diam } Q')^{2-p} + \sigma_p(Q'; \mathcal{K})\} \{(\text{diam } Q'')^{2-p} + \sigma_p(Q''; \mathcal{K})\}} \leq \lambda$$

holds.

The constants of equivalence in (1.2) depend only on p .

Remark 1.3 For the peace of mind of any particularly pedantic reader, we mention that ultimately we can choose the constant γ to equal, for example, 2^{22} . \triangleleft

The trace criterion given in this theorem describes the structure of the trace space $L_p^2(\mathbf{R}^2)|_E$ and shows which properties of a function f on E control its almost optimal extension to a function from $L_p^2(\mathbf{R}^2)$. At the same time it is not quite clear how one could check the conditions of part (i) and part (ii) of Theorem 1.2 for a given function f on E . In fact, these conditions depend on an infinite number of families of squares (\mathcal{Q}, \mathcal{K} etc.), triangles and triples of collinear points.

Nevertheless a careful examination of our proof shows that it constructs *four particular families* of squares, triangles and points, etc., depending only on the set E and that it is enough to examine the behavior of given functions f only on these particular families.

It is convenient to express this fact by the following theorem, which refines one part of Theorem 1.2. We are very grateful to Charles Fefferman for conjecturing and motivating us to seek a result along these lines.

Theorem 1.4 Let $2 < p < \infty$ and let E be a finite subset of \mathbf{R}^2 . There exist absolute constants $\gamma > 0$, $C > 0$ and $N \in \mathbf{N}$ and:

(i) A family $\{Q_i : i = 1, \dots, m\}$ with $m \leq C \#E$ of pairwise disjoint squares and a family

$$\{z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \in E \cap (\gamma Q_i) : z_2^{(i)} \in (z_1^{(i)}, z_3^{(i)}), i = 1, \dots, m\}$$

of triples of collinear points;

(ii) A family \mathcal{K} of pairwise disjoint squares with $\#\mathcal{K} \leq C \#E$, and a mapping

$$\mathcal{K} \ni K \mapsto \Delta(K) \in \text{Triangle}(E)$$

such that $\Delta(K) \subset \gamma K$ and $\text{diam } K \leq \gamma \text{diam } \Delta(K)$ for every $K \in \mathcal{K}$;

(iii) A family of squares \mathcal{Q} with covering multiplicity $M(\mathcal{Q}) \leq N$ and $\#\mathcal{Q} \leq C \#E$, and mappings $\mathcal{Q} \ni Q \mapsto Q' \in \mathcal{Q}$ and $\mathcal{Q} \ni Q \mapsto Q'' \in \mathcal{Q}$ satisfying the condition $Q' \cup Q'' \subset \gamma Q$ for all $Q \in \mathcal{Q}$,

such that for every function $f : E \rightarrow \mathbf{R}$ the following equivalence

$$\begin{aligned} \|f\|_{L_p^2(\mathbf{R}^2)|_E} &\sim \left(\sum_{i=1}^m \left| \frac{f(z_1^{(i)}) - f(z_2^{(i)})}{\|z_1^{(i)} - z_2^{(i)}\|_2} - \frac{f(z_2^{(i)}) - f(z_3^{(i)})}{\|z_2^{(i)} - z_3^{(i)}\|_2} \right|^p (\text{diam } Q_i)^{2-p} \right)^{\frac{1}{p}} \\ &+ \left(\sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} S_p(f : Q', Q''; \mathcal{K})}{\{(\text{diam } Q')^{2-p} + \sigma_p(Q'; \mathcal{K})\} \{(\text{diam } Q'')^{2-p} + \sigma_p(Q''; \mathcal{K})\}} \right)^{\frac{1}{p}} \end{aligned}$$

holds. The constants of this equivalence depend only on p .

A structure of the trace norms of functions defined on a finite set $E \subset \mathbf{R}^n$ in the space $L_p^m(\mathbf{R}^n)|_E$ has been discussed and studied by A. Israel in [24] and by C. Fefferman, A. Israel and G. K. Luli in [20]. In these papers the authors introduce a notion of a linear functional with the so-called “assisted bounded depth” and characterize the trace norm of a function on E using this notion. They also formulate several open problems related to the structure of the trace norm. Let us formulate one of them.

Problem 1.5 (A. Israel [24]) *Given finite set $E \subset \mathbf{R}^2$ do there exist linear functionals*

$$\{\lambda_i\}_{i=1}^L \subset (L_p^2(\mathbf{R}^2)|_E)^*, \quad L \leq C \#E,$$

each depending only on N values such that

$$\|f\|_{L_p^2(\mathbf{R}^2)|_E}^p \sim \sum_{i=1}^L |\lambda_i(f)|^p$$

for every function $f : E \rightarrow \mathbf{R}$?

Here $N > 0$ is a certain absolute constant.

Our next result provides a solution to this problem with $N = 6$ whenever $2 < p < \infty$.

Theorem 1.6 *Let $2 < p < \infty$ and let $E \subset \mathbf{R}^2$ be a finite set. There exist linear functionals*

$$\{\lambda_1, \lambda_2, \dots, \lambda_L\}, \quad L \leq C \#E,$$

each depending on at most six values of a function on E such that for every function $f : E \rightarrow \mathbf{R}$ the following equivalence

$$\|f\|_{L_p^2(\mathbf{R}^2)|_E}^p \sim \sum_{i=1}^L |\lambda_i(f)|^p$$

holds.

The constants of this equivalence depend only on p .

The proof of this theorem and Theorem 1.4, including an explicit construction of the objects which they mention and use, are given in Section 11. Note that the objects described in parts (i), (ii) and (iii) of Theorem 1.4 and the linear functionals $\{\lambda_i\}$ from Theorem 1.6 depend only on E and p .

In the next four remarks we will briefly review several previous results about Sobolev extensions which are related to Theorem 1.2.

Remark 1.7 The case $p = \infty$ was studied in the author's papers [32, 33, 34, 36]. (Recall that $L_\infty^2(\mathbf{R}^2)$ coincides with the space $C^{1,1}(\mathbf{R}^2)$ of all C^1 -functions whose partial derivatives of the first order are Lipschitz continuous on \mathbf{R}^2 .) In those papers we proved that a function f defined on an arbitrary closed set $E \subset \mathbf{R}^2$ extends to a function $F \in L_\infty^2(\mathbf{R}^2)$ if and only if there exists a constant $\lambda > 0$ such that:

(a). For every three collinear points $z_1, z_2, z_3 \in E$ such that $z_2 \in (z_1, z_3)$ the following inequality

$$\left| \frac{f(z_1) - f(z_2)}{\|z_1 - z_2\|_2} - \frac{f(z_2) - f(z_3)}{\|z_2 - z_3\|_2} \right| \leq \lambda \|z_1 - z_3\|_2$$

holds;

(b). For every two triangles Δ_1 and Δ_2 with vertices in E we have

$$\|\nabla P_{\Delta_1}[f] - \nabla P_{\Delta_2}[f]\| \leq \lambda \{R_{\Delta_1} + R_{\Delta_2} + \text{diam}(\Delta_1 \cup \Delta_2)\}.$$

Furthermore, $\|f\|_{L_\infty^2(\mathbf{R}^2)|_E} \sim \inf \lambda$.

Note that in parts (a) and (b) of this criterion we use the values of the function f at no more than 6 points of the set E . This phenomenon is a consequence of the so-called “finiteness principle” for the space $C^{1,1}(\mathbf{R}^2)$ proven in [33, 34]. (See also [7].) This principle provides a criterion for calculation of the trace norm $\|f\|_{C^{1,1}(\mathbf{R}^2)|_E}$ in the following form: Let $N = 6$. Then

$$\|f\|_{C^{1,1}(\mathbf{R}^2)|_E} \sim \sup \{\|f|_{E'}\|_{C^{1,1}(\mathbf{R}^2)|_{E'}} : E' \subset E, \#E' \leq N\}.$$

The same formula is true for the space $L_\infty^2(\mathbf{R}^n) = C^{1,1}(\mathbf{R}^n)$ with $N = 3 \cdot 2^{n-1}$ (and this value is sharp), see [34].

The finiteness principle for the space $L_\infty^m(\mathbf{R}^n) = C^{m-1,1}(\mathbf{R}^n)$ with arbitrary m and n and a certain constant $N = N(m, n)$ has been proved by C. Fefferman [9]. \triangleleft

Remark 1.8 Whitney-type extension problems for the Sobolev space $L_p^1(\mathbf{R}^n)$ have been studied in the author's papers [39, 41]. We have proved that *the classical extension operator constructed by Whitney in [44] for the space $C^1(\mathbf{R}^n)$ provides an almost optimal extension operator for the trace space $L_p^1(\mathbf{R}^n)|_E$ whenever $n < p < \infty$ and E is an arbitrary closed subset of \mathbf{R}^n* . This enables us to give several constructive trace criteria. Let us mention two of them here.

In particular, a function $f : E \rightarrow \mathbf{R}$ can be extended to a (continuous) function $F \in L_p^1(\mathbf{R}^n)$, $n < p < \infty$, if and only if there exists a constant $\lambda > 0$ such that for every finite family $\{Q_i : i = 1, \dots, m\}$ of pairwise disjoint cubes in \mathbf{R}^n and every choice of points $x_i, y_i \in (11Q_i) \cap E$ the following inequality

$$\sum_{i=1}^m \frac{|f(x_i) - f(y_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq \lambda$$

holds. Furthermore, $\|f\|_{L_p^1(\mathbf{R}^n)|_E} \sim \inf \lambda^{\frac{1}{p}}$.

The second criterion is expressed as an explicit formula for the order of magnitude of the trace norm of a function $f : E \rightarrow \mathbf{R}$ in the space $L_p^1(\mathbf{R}^n)$ whenever $p \in (n, \infty)$:

$$(1.6) \quad \|f\|_{L_p^1(\mathbf{R}^n)|_E} \sim \left\{ \int_{\mathbf{R}^n} \left(\sup_{y, z \in E} \frac{|f(y) - f(z)|}{\|x - y\| + \|x - z\|} \right)^p dx \right\}^{\frac{1}{p}}.$$

See [39], Theorem 1.4, and [41], Theorem 6.1.

In both of these criteria the constants of the equivalences depend only on n and p .

We remark that in [39] we also present a series of trace criteria expressed in terms of local oscillations of functions with respect to certain *doubling measures* supported on E . These results have been inspired by a paper of A. Jonsson [25]. In particular, in [25] A. Jonsson presents a trace criterion of a similar kind which provides a constructive characterization of the restrictions of Besov spaces $B_{p,q}^\alpha(\mathbf{R}^n)$ to an arbitrary closed set $E \subset \mathbf{R}^n$ whenever $0 < \alpha < 1$ and $p\alpha > n$. \triangleleft

Remark 1.9 Let $\{P_x \in \mathcal{P}_{m-1}(\mathbf{R}^n) : x \in E\}$ be a polynomial field, i.e., a mapping which to every point $x \in E$ assigns a polynomial P_x on \mathbf{R}^n of degree at most $m - 1$. Given an m -times differentiable function F and a point $x \in \mathbf{R}^n$, we let $T_x^m[F]$ denote the Taylor polynomial of F at x of order m . We recall one of Whitney classical extension theorems [44] which states that

$$\begin{aligned} & \inf \left\{ \|F\|_{L_\infty^m(\mathbf{R}^n)} : F \in L_\infty^m(\mathbf{R}^n), T_x^{m-1}[F] = P_x \text{ for every } x \in E \right\} \\ & \sim \sum_{|\alpha| \leq m-1} \sup_{y, z \in E} |D^\alpha(P_y - P_z)(y)| / \|y - z\|^{m-|\alpha|}. \end{aligned}$$

See also Glaeser [23].

In [41] we generalize this theorem to the case of the Sobolev space $L_p^m(\mathbf{R}^n)$, $p > n$, proving that for every polynomial field $\{P_x \in \mathcal{P}_{m-1}(\mathbf{R}^n) : x \in E\}$ the following equivalence

$$\inf \left\{ \|F\|_{L_p^m(\mathbf{R}^n)} : F \in L_p^m(\mathbf{R}^n), T_x^{m-1}[F] = P_x \text{ for every } x \in E \right\} \\ \sim \sum_{|\alpha| \leq m-1} \left(\int_{\mathbf{R}^n} \left(\sup_{y, z \in E} \frac{|D^\alpha(P_y - P_z)(y)|}{\|x - y\|^{m-|\alpha|} + \|x - z\|^{m-|\alpha|}} \right)^p dx \right)^{\frac{1}{p}}$$

holds with constants depending only on m, p , and n . Clearly, for $m = 1$ we obtain (1.6).

As in the case $m = 1$ (see Remark 1.8) we also show in [41] that the Whitney extension operator constructed in [44] for the space $C^m(\mathbf{R}^n)$ provides an almost optimal extension for every polynomial field $\{P_x \in \mathcal{P}_{m-1}(\mathbf{R}^n) : x \in E\}$ on E whenever $n < p < \infty$ and E is an arbitrary closed subset of \mathbf{R}^n . \triangleleft

Remark 1.10 In [41] we study a Whitney-type extension problem for the space $L_p^m(\mathbf{R})$, $p > 1$. Let $E \subset \mathbf{R}$ be an arbitrary closed set and let T_E be the extension operator constructed by Whitney in [45] for the trace space $L_\infty^m(\mathbf{R})|_E = C^{m-1,1}(\mathbf{R})|_E$. We show that this very same Whitney extension operator T_E also provides an almost optimal extension of each function $f \in L_p^m(\mathbf{R})|_E$ whenever $1 < p < \infty$.

This leads us to an analogue of Whitney trace criterion [45] for the space $L_\infty^m(\mathbf{R})$. (See also J. Merrien [29].) We recall that this classical result states that, for arbitrary positive integer m and every function f on E ,

$$\|f\|_{L_\infty^m(\mathbf{R})|_E} \sim \sup_{S \subset E, \#S=m+1} |\Delta^m f[S]|.$$

In the preceding formula $\Delta^m f[S]$ denotes *the divided difference* of f on S , i.e., for each finite set $S = \{x_0, x_1, \dots, x_m\} \subset \mathbf{R}$, it is given by

$$\Delta^m f[S] = \Delta^m f[x_0, x_1, \dots, x_m] = \sum_{i=0}^m \frac{f(x_i)}{\prod_{0 \leq k \leq m, k \neq i} (x_i - x_k)}.$$

In [41] we find a counterpart of the preceding result, in which we replace L_∞^m by L_p^m . We prove that for arbitrary $1 < p < \infty$ and every function f on E

$$\|f\|_{L_p^m(\mathbf{R})|_E} \sim \left\{ \int_{\mathbf{R}} \sup_{S \subset E, \#S=m+1} \left(\frac{|\Delta^m f[S]| \operatorname{diam} S}{\operatorname{diam}(\{x\} \cup S)} \right)^p dx \right\}^{\frac{1}{p}} \\ \sim \left\{ \int_{\mathbf{R}} \sup_{x_0 < x_1 < \dots < x_m, x_i \in E} \frac{|\Delta^{m-1} f[x_0, \dots, x_{m-1}] - \Delta^{m-1} f[x_1, \dots, x_m]|^p}{|x - x_0|^p + |x - x_m|^p} dx \right\}^{\frac{1}{p}}$$

with constants of equivalence depending only on p and m . \triangleleft

Our next result, Theorem 1.11, provides a slight modification of the trace criterion given in Theorem 1.2. More specifically, part (ii) of the new criterion is expressed in terms of families of *disks* with certain geometric constraints on the curvature of their boundaries.

Before we formulate Theorem 1.11 let us define several additional notions and fix additional notation. For a disk D we let c_D and r_D denote its center and radius respectively. By \mathbf{c}_D we denote the curvature of the boundary of the disk D , i.e., the reciprocal of its radius:

$$(1.7) \quad \mathbf{c}_D = \frac{1}{r_D}.$$

We refer to \mathbf{c}_D as *the curvature of the disk D* .

Let $p \in [1, \infty)$. Let \mathcal{D} be a family of pairwise disjoint disks in \mathbf{R}^2 and let Δ be a mapping which to every disk $D \in \mathcal{D}$ assigns a triangle $\Delta(D) \subset \mathbf{R}^2$.

Given a disk $B \subset \mathbf{R}^2$ we define a quantity $\mathbf{c}_p(B : \mathcal{D}, \Delta)$ by the formula:

$$\mathbf{c}_p(B : \mathcal{D}, \Delta) := \left\{ \frac{1}{|B|} \sum_{D \in \mathcal{D}} \mathbf{c}_{\Delta(D)}^p |D| \right\}^{\frac{1}{p}}.$$

We refer to $\mathbf{c}_p(B : \mathcal{D}, \Delta)$ as *the p -average of the Menger curvature on B with respect to the family \mathcal{D} and the mapping Δ* .

Theorem 1.11 *Let $2 < p < \infty$. Let E be a finite subset of \mathbf{R}^2 and let f be a function defined on E . Then*

$$(1.8) \quad \|f\|_{L_p^2(\mathbf{R}^2)|_E} \sim \inf \lambda^{\frac{1}{p}}$$

where the infimum is taken over all positive constants λ which satisfy all of the following conditions for a certain absolute positive constant γ :

(i). *The condition (i) of Theorem 1.2 holds;*

(ii). *Let \mathcal{B} and \mathcal{D} be arbitrary finite families of pairwise disjoint disks. Suppose that to each disk $D \in \mathcal{D}$ we have arbitrarily assigned a triangle $\Delta(D)$ in E such that*

$$\Delta(D) \subset \gamma D \quad \text{and} \quad \text{diam } D \leq \gamma \text{diam } \Delta(D).$$

Suppose that to each disk $B \in \mathcal{B}$ we have arbitrarily assigned two disks $B', B'' \in \mathcal{B}$ such that $B' \cup B'' \subset \gamma B$ and

$$(1.9) \quad \mathbf{c}_p(B' : \mathcal{D}, \Delta) \leq \mathbf{c}_{B'} \quad \text{and} \quad \mathbf{c}_p(B'' : \mathcal{D}, \Delta) \leq \mathbf{c}_{B''}.$$

Let

$$S_p(f : B', B''; \mathcal{D}) := \sum_{\substack{D' \in \mathcal{D} \\ c_{D'} \in B'}} \sum_{\substack{D'' \in \mathcal{D} \\ c_{D''} \in B''}} \|\nabla P_{\Delta(D')}[f] - \nabla P_{\Delta(D'')}[f]\|^p \mathbf{c}_{\Delta(D')}^p |D'| \mathbf{c}_{\Delta(D'')}^p |D''|.$$

Then the following inequality

$$\sum_{B \in \mathcal{B}} \left(\frac{\text{diam } B' \text{diam } B''}{\text{diam } B} \right)^{p-2} S_p(f : B', B''; \mathcal{D}) \leq \lambda$$

holds.

The constants of equivalence in (1.8) depend only on p .

We prove this theorem in Section 11.

The proof of Theorem 1.2 uses the extension methods developed for the Sobolev space $L_\infty^2(\mathbf{R}^n) = C^{1,1}(\mathbf{R}^n)$, see [34, 36]. It also uses Theorem 3.7 about sums of Sobolev and L_p -weighted spaces. (See also the discussion in Section 9.)

The methods of the proof of Theorem 1.2 and some related ideas enable us to construct an almost optimal extension algorithm which *linearly* depends on a function from the trace space $L_p^2(\mathbf{R}^2)|_E$. Thus we can give a new and different proof of the following recent theorem of Arie Israel [24].

Theorem 1.12 *For every finite subset $E \subset \mathbf{R}^2$ and every $p > 2$ there exists a linear extension operator which maps the trace space $L_p^2(\mathbf{R}^2)|_E$ continuously into $L_p^2(\mathbf{R}^2)$. Its operator norm is bounded by a constant depending only on p .*

It seems worthwhile at this point to briefly describe and compare the methods used in these two different proofs. (Both A. Israel and the author initially presented details of their respective approaches to this result at the workshop “Differentiable structures on finite sets” organized by American Institute of Mathematics, Palo Alto, CA, in 2010.)

In [24] A. Israel generalizes and introduces new elements into an approach developed earlier by C. Fefferman [9, 12] in his proof of the “finiteness principle” for the trace space $L_\infty^m(\mathbf{R}^n)|_E = C^{m-1,1}(\mathbf{R}^n)|_E$. (See also [11, 14, 16].) One of the main ingredients of this approach is an elaborate Calderón-Zygmund type decomposition of \mathbf{R}^2 into appropriate “Calderón-Zygmund” cubes each containing a portion of E which is in a certain sense “convenient” for extension. In Fefferman’s earlier proof, which deals with the case $p = \infty$, each of the “Calderón-Zygmund” cubes are treated on an “equal footing”; no cube has a more important role than any of the other cubes. However, in [24], for the case $p < \infty$, a variant of the above-mentioned decomposition yields cubes, in fact squares since $n = 2$, which are no longer all on an equal footing. A certain special subfamily of them, which are referred to as Keystone squares, have an important role, and all the others can be ignored. The Keystone squares concentrate all information which is necessary for an almost optimal extension of a function defined on E to a function from the Sobolev space $L_p^2(\mathbf{R}^2)$.

Let us now turn to the author’s different approach to the proof of Theorem 1.12 whose details appear below in Section 2. The reader will be able to note some features which are common to both approaches. However, the starting point here comes from a method developed and used earlier in the papers [32, 34, 36] where the author proved the above-mentioned finiteness principle for the space $L_\infty^2(\mathbf{R}^2) = C^{1,1}(\mathbf{R}^2)$. (Cf. Remark 1.7.) After modifying this method to cope with the case $p < \infty$ we are led to a new problem of constructing an almost optimal decomposition of a given function into a sum of a Sobolev and a weighted L_p -function. The particular weight function which must be used here is generated by the Menger curvature of certain triangles with vertices in E . The solution of a rather more general version of this problem (for an arbitrary weight function or measure) is presented separately in [40]. The main tool there is a filtering procedure which determines a special family of squares in \mathbf{R}^2 (we call them “important squares”). These squares possess certain measure concentration properties and together contain all necessary information about extension properties of a function defined on E .

At this stage we do not know if there is any relation between the Keystone squares introduced in [24] and the “important squares” with respect to the Menger curvature

measure which are treated in this paper. Perhaps, for a given set E they even coincide exactly with each other. We share the opinion expressed in [24] and [20] that it would be interesting and useful to investigate possible connections between the Keystone squares and the “important” squares.

Let us mention some other results which are related to Theorem 1.12. We note that the existence of a continuous linear extension operator for the space $L_\infty^2(\mathbf{R}^n) = C^{1,1}(\mathbf{R}^n)$ was shown by Yu. Brudnyi and the author [6]. C. Fefferman [11] proved that such an operator exists for the space $L_\infty^m(\mathbf{R}^n)$ for arbitrary m .

Quite recently C. Fefferman, A. Israel and G. K. Luli [20] proved that a continuous linear extension operator exists for the trace space $L_p^m(\mathbf{R}^n)|_E$ whenever $n < p < \infty$ and $E \subset \mathbf{R}^n$ is an arbitrary closed set. This remarkable result also relies on ideas and techniques related to the above mentioned Calderón-Zygmund type decomposition and certain properties of Keystone cubes.

Finally we note that G. K. Luli [26] proved the existence of a continuous linear extension operator for the space $L_p^m(\mathbf{R})|_E$, $p > 1$, for every *finite* set $E \subset \mathbf{R}$. As we have noted in Remark 1.10 the extension operator T_E constructed by H. Whitney [45] for the space $C^m(\mathbf{R})|_E$ provides an almost optimal extension for every trace space $L_p^m(\mathbf{R})|_E$ and *every* closed set $E \subset \mathbf{R}$ whenever $p > 1$. (See [41].) Since T_E is *linear*, this proves the existence of a continuous linear extension operator $L_p^m(\mathbf{R})|_E$, $p > 1$, for an *arbitrary* closed set $E \subset \mathbf{R}$.

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2. Plan of the proof of the Main Theorem 1.2.

Let us briefly describe the main stages of the proof of Theorem 1.2.

In **Section 3** we prove that for every function f on E the conditions (i) and (ii) of this theorem are satisfied with $\lambda = C\|f\|_{L_p^2(\mathbf{R}^2)|_E}^p$ where C is a constant depending only on p . We refer to this part of the proof of Theorem 1.2 as the “necessity” part.

More specifically, we fix a constant $\gamma \in [1, \infty)$ and a function $F \in L_p^2(\mathbf{R}^2)$, $p > 2$, such that $F|_E = f$ and show that the statements (i) and (ii) of Theorem 1.2 hold with $\lambda = C(p, \gamma)\|F\|_{L_p^2(\mathbf{R}^2)}^p$. The proof of these statements is based on the classical Sobolev-Poincaré inequality for Sobolev functions which we recall in Proposition 3.2. Let us give some remarks related to the proof of part (ii); the part (i) is much more simpler and its proof follows from the Sobolev-Poincaré inequality and the Hardy-Littlewood maximal theorem.

The point of departure of our proof of part (ii) is Proposition 3.5. Given a square Q , a triangle $\Delta \subset Q$ and a point $x \in Q$ this proposition provides a bound of the distance between $\nabla F(x)$ and $\nabla P_\Delta[F]$. Recall that $P_\Delta[F]$ is the affine polynomial which interpolates F at the vertices of Δ . This bound depends on the radius R_Δ of the circle passing through

the vertices of Δ . Recall that the Menger curvature $\mathbf{c}_\Delta = 1/R_\Delta$ so that \mathbf{c}_Δ is implicitly involved in this bound.

Let \mathcal{K} be a family of pairwise disjoint squares and let $\mathcal{C}_\mathcal{K} = \{c_K : K \in \mathcal{K}\}$ be the family of its centers. Given $K \in \mathcal{K}$ let $\Delta(K)$ be a triangle satisfying condition (1.4) of Theorem 1.2. Proposition 3.5 motivates us to introduce two important objects - a mapping $\mathcal{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ and a non-negative Borel measure μ on \mathbf{R}^2 . The mapping \mathcal{T} is supported on the set $\mathcal{C}_\mathcal{K}$ and for each square $K \in \mathcal{K}$ it takes the value $\mathcal{T}(c_K) := \nabla P_{\Delta(K)}[F]$ at the center of K . The measure μ is a discrete measure supported on $\mathcal{C}_\mathcal{K}$ such that $\mu(\{c_K\}) = \mathbf{c}_{\Delta(K)}^p |K|$ for each $K \in \mathcal{K}$.

Using Proposition 3.5 and the Hardy-Littlewood maximal theorem we prove that the mapping \mathcal{T} belongs to the sum $\vec{\Sigma}$ of the vector Sobolev space $\vec{L}_p^1(\mathbf{R}^2)$ and the vector L_p -space $\vec{L}_p(\mathbf{R}^2; \mu)$ with respect to the measure μ . Furthermore, we show that $\|\mathcal{T}\|_{\vec{\Sigma}} \leq C(p)\|F\|_{L_p^2(\mathbf{R}^2)}$. This is a crucial point of the proof of the necessity.

In [40] we present a general formula for calculation of the norm of a function in the sum of the Sobolev space $L_p^1(\mathbf{R}^2)$, $p > n$, and a space $L_p(\mathbf{R}^n; \mu)$ where μ is an arbitrary non-negative Borel measure on \mathbf{R}^n , see Theorem 3.7. Applying this theorem to the mapping \mathcal{T} we immediately obtain the required statement of part (ii) of Theorem 1.2.

For the reader's convenience, we also give a direct proof of this statement which does not use results of the work [40]. This proof relies only on the classical Hardy-Littlewood maximal theorem and Proposition 3.6 which provides a Sobolev-Poincaré type inequality for pairs of interpolating affine polynomials in \mathbf{R}^2 .

In Sections 4-8 we prove that if a function $f : E \rightarrow \mathbf{R}$ and a constant $\lambda > 0$ satisfy conditions (i) and (ii) of Theorem 1.2, then $\|f\|_{L_p^2(\mathbf{R}^2)|_E}^p \leq C(p)\lambda$. We refer to this part of the proof of Theorem 1.2 as the “sufficiency” part. Let us describe its main stages.

Sections 4-6 of the paper deal with some geometric preparations which are basic elements of our construction of an extension operator for the Sobolev space $L_p^2(\mathbf{R}^2)$. The main goal of these preparations is to define a certain kind of geometric “structure” within the set E which will enable us to organize and characterize certain “holes” in the complement $\mathbf{R}^2 \setminus E$ and to characterize certain kinds of “contacts” between them. We refer to these “holes” as “lacunae”. Although in this paper we will only be dealing with subsets of \mathbf{R}^2 , it turns out to be just as easy to present these geometric preparations in the case of subsets of \mathbf{R}^n for arbitrary n . Therefore the material of Sections 4-6 deals with the general n -dimensional case, with a view towards possible future applications of this approach. Later on, in **Section 7**, we go back to dealing only with the two dimensional case.

Let $E \subset \mathbf{R}^n$ be an arbitrary closed subset. We let W_E denote a *Whitney covering* of the open set $\mathbf{R}^n \setminus E$, i.e., a family of non-overlapping cubes (so-called “Whitney cubes”) such that for each $Q \in W_E$ we have $\text{diam } Q \sim \text{dist}(Q, E)$. (See Theorem 4.1.) In Section 4 we introduce and study one of the main ingredients of our approach - the notion of a *lacuna* of Whitney cubes with respect to E .

Roughly speaking a lacuna L is a “hole” in the set $\mathbf{R}^n \setminus E$, a collection of cubes in W_E which surrounds a certain small subset V_L of the set E . L is defined by the two requirements that

$$(10Q) \cap E = (90Q) \cap E \quad \text{for each } Q \in L$$

and

$$(10Q) \cap E = (10Q') \cap E = V_L \text{ for each } Q \text{ and } Q' \text{ in } L.$$

We let \mathcal{L}_E denote the set of all lacunae with respect to E .

In Section 4 we establish several important geometric properties of lacunae. In particular, we show that, for each lacuna L , the diameter of the associated set V_L is equivalent to the diameter of the minimal cube of L . (See Proposition 4.4.) We also show that the diameter of the maximal cube of L is equivalent to the diameter of the union of all the cubes of L . (See Proposition 4.7.)

In **Section 5** we continue the study of geometric properties of lacunae. In particular, in Proposition 5.3, we show that there exists a mapping $\mathcal{L}_E \ni L \mapsto \mathcal{PR}(L) \in E$ such that:

- (i) $\mathcal{PR}(L)$ lies in a fixed dilation of the minimal cube of the lacuna, and
- (ii) every point $A \in E$ has at most $C(n)$ “sources”, i.e., lacunae $L' \in \mathcal{L}_E$ such that $\mathcal{PR}(L') = A$.

We refer to the mapping \mathcal{PR} as a “projection” of \mathcal{L}_E into the set E . In particular, the existence of a projection enables us to show that for each finite set E the number of its lacunae does not exceed $C(n)$ times the number of points in E . (See Corollary 5.4.)

In this section we also introduce a special graph \mathcal{G} whose vertices are all the lacunae of the set E . We define the edges of \mathcal{G} by saying that two lacunae $L, L' \in \mathcal{L}_E$ are joined by an edge if there exist cubes $Q \in L, Q' \in L'$ such that $Q \cap Q' \neq \emptyset$. In this case we call the cubes Q, Q' *contacting cubes* and the lacunae L, L' *contacting lacunae*. We use the notation $L \leftrightarrow L'$ to denote that L and L' are contacting.

The above-mentioned properties of lacunae imply certain special properties of the graph \mathcal{G} . In particular, we prove that the degree of every vertex of \mathcal{G} is bounded by a constant $C(n)$; furthermore, the number of vertices does not exceed $C(n)$ times the number of points in E . We also show that if a cube $Q \in L$ intersects some cube Q' of some other lacuna, (i.e. if Q and Q' are contacting cubes) then either $\text{diam } Q$ is almost minimal or almost maximal in the lacuna L .

For each lacuna L we need to choose two points A_L and B_L in E which will play an important role in the sequel. If V_L is a single point, we choose A_L to be that point, and we may choose B_L to be some point in $E \setminus \{A_L\}$ whose distance from A_L is minimal. Otherwise we may choose A_L and B_L to be any pair of distinct points in V_L which satisfy $\|A_L - B_L\| = \text{diam } V_L$. We refer to the ordered pair (A_L, B_L) as an *interior bridge* of the lacuna L . We call the points A_L and B_L *the ends of the interior bridge*.

We note that the fact that two lacunae L and L' are contacting, does not guarantee any connection between their respective interior bridges.

In **Section 6** we construct additional bridges which join interior bridges between contacting lacunae. We refer to these new bridges as *exterior bridges*. More specifically, given contacting lacunae L and L' we choose points $C(L, L') \in \{A_L, B_L\}$ and $C(L', L) \in \{A_{L'}, B_{L'}\}$ in such a way that the following two conditions are satisfied:

In the triangle $\Delta\{A_L, B_L, C(L', L)\}$ the side of the triangle which is opposite to the vertex $C(L, L')$ is not the smallest side of the triangle.

A similar condition holds for the triangle $\Delta\{A_{L'}, B_{L'}, C(L, L')\}$: the side of the triangle which is opposite to the vertex $C(L', L)$ is not the smallest side of the triangle.

We refer to the (non-ordered) pair $\{C(L, L'), C(L', L)\}$ as an *exterior bridge connecting the interior bridges* (A_L, B_L) and $(A_{L'}, B_{L'})$. We prove that for every pair of contacting lacunae L and L' there always exists an exterior bridge which connects L to L' . We say that the interior bridge $T = (A_L, B_L)$ and the exterior bridge $\tilde{T} = \{C(L, L'), C(L', L)\}$ are *connected bridges*. In this case we write $T \rightsquigarrow \tilde{T}$.

This special choice of the ends $C(L, L')$ and $C(L', L)$ of the exterior bridge implies a certain special property of the triangle $\Delta = \Delta\{A_L, B_L, C(L', L)\}$. We know that $C(L, L')$ is one of the vertices of Δ . This point is a common point of the interior bridge (A_L, B_L) and the exterior bridge $\{C(L, L'), C(L', L)\}$. Let α be the angle of the triangle Δ corresponding to the vertex $C(L, L')$. Then

$$|\sin \alpha| \sim \frac{1}{R_\Delta} \text{diam } \Delta = \mathbf{c}_\Delta \text{diam } \Delta.$$

We use this specific property of the triangle Δ later in our estimates of the Sobolev norm of the extension operator. In particular, it explains the appearance of the Menger curvature \mathbf{c}_Δ in the criterion of Theorem 1.2.

By \mathcal{BR}_E we denote the family of all bridges (interior and exterior). For every bridge $T \in \mathcal{BR}_E$ we let $A^{[T]}$ and $B^{[T]}$ denote its ends. We know that some pairs $T, T' \in \mathcal{BR}_E$ of the bridges are connected to each other ($T \rightsquigarrow T'$). We also know that one of these connected bridges is an interior bridge of a lacuna, and the second one is an exterior bridge joining L with another contacting lacuna.

We finish our geometric preparations with the following statement proven in Proposition 6.6:

There exists a family \mathcal{K}_E of pairwise disjoint cubes and a one-to-one mapping which to every pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \rightsquigarrow T'$, assigns a cube $K(T, T') \in \mathcal{K}_E$ such that $\text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \sim \text{diam } K(T, T')$ and $\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \subset \gamma K(T, T')$.

In **Section 7**, as already mentioned, we return to dealing only with the case where the dimension $n = 2$. We turn to the construction of an extension operator for the trace space $L_p^2(\mathbf{R}^2)|_E$ whenever $E \subset \mathbf{R}^2$ is an arbitrary *finite* subset and $p > 2$.

We fix a function $f : E \rightarrow \mathbf{R}$. To every lacuna $L \in \mathcal{L}_E$ we are going to assign an affine polynomial $P_L \in \mathcal{P}_1$ which *interpolates f at the points A_L and B_L* . Note that we have some flexibility (one degree of freedom) in the choice of this polynomial. A collection of several results, in Section 7 and also in Section 8, ultimately enable us to develop an appropriate strategy for choosing P_L in an “almost optimal” way.

Once we have obtained a suitable family of affine polynomials $\{P_L \in \mathcal{P}_1 : L \in \mathcal{L}_E\}$ by this procedure, we can use it to generate a Whitney-type extension of f to all of \mathbf{R}^2 in the following way: First, we assign a polynomial $P^{(Q)}$ to every Whitney square $Q \in W_E$. We do this by setting $P^{(Q)} := P_L$ for all the squares Q in the lacuna L , for each $L \in \mathcal{L}_E$. Then we define an extension F of f by the Whitney formula

$$(2.1) \quad F(x) := \sum_{Q \in W_E} \varphi_Q(x) P^{(Q)}(x), \quad \text{whenever } x \in \mathbf{R}^2 \setminus E,$$

and $F(x) = f(x)$, if $x \in E$. Here $\{\varphi_Q : Q \in W_E\}$ is a smooth partition of unity subordinated to the Whitney decomposition W_E with standard properties (as described in Lemma 7.2).

Now let us begin sketching the (rather long) series of results which we need to subsequently fix our strategy for choosing the affine polynomials P_L . Our goal will be to choose them in a way that ensures that the function F defined by (2.1) satisfies

$$(2.2) \quad \|F\|_{L_p^1(\mathbf{R}^2)} \leq C(p)\lambda^{\frac{1}{p}}$$

whenever the constant λ is related to the function f by the conditions (i), (ii) of Theorem 1.2. This will of course complete the proof of the sufficiency part of Theorem 1.2.

Our first step is to show that, no matter how we fix the remaining degree of freedom in our choice of the polynomials P_L , the function F of (2.1) will always satisfy

$$\|F\|_{L_p^2(\mathbf{R}^2)}^p \leq C \sum d(L, L')^{2-2p} \max_{Q(L, L')} |P_L - P_{L'}|^p$$

where the sum is taken over all pairs of *contacting* lacunae $L, L' \in \mathcal{L}_E$ ($L \leftrightarrow L'$). Here $d(L, L') := \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\}$, and $Q(L, L') := Q(A_L, d(L, L'))$. (See Proposition 7.3).

In the next step, in addition to the polynomials P_L (which interpolate f at the ends of the interior bridges (A_L, B_L)) we also bring into play other affine polynomials which interpolate f at the ends of *exterior* bridges. Our aim is to obtain an estimate for the norm $\|F\|_{L_p^2(\mathbf{R}^2)}^p$ of the function F of (2.1), in terms of the *gradients* of all these affine polynomials. As in the previous step, the result here will hold no matter how we fix the remaining degree of freedom in the choice of the polynomials.

It is convenient to consider “gradient” mappings $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ which assign a vector in \mathbf{R}^2 to each (interior or exterior) bridge $T \in \mathcal{BR}_E$. We note that such a mapping g satisfies

$$(2.3) \quad \langle g(T), A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]})$$

if and only if it is the gradient of an affine polynomial which interpolates f at the ends $A^{[T]}$ and $B^{[T]}$ of the bridge T . (Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbf{R}^2 .)

Suppose that the mapping g satisfies (2.3) for *every* bridge $T \in \mathcal{BR}_E$ with ends at points $A^{[T]}, B^{[T]} \in E$. Then we prove (see Proposition 7.4) that, if we take the polynomials P_L to be $P_L(x) = f(A_L) + \langle g(T_L), x - A_L \rangle$ where $T_L = (A_L, B_L)$ is the interior bridge of the lacuna L , then the function F of (2.1) satisfies

$$\|F\|_{L_p^2(\mathbf{R}^2)}^p \leq C \sum \{D(T, T')^{2-p} \|g(T) - g(T')\|^p : T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'\}.$$

Here $D(T, T') := \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}$ for each pair $T, T' \in \mathcal{BR}_E$ of connected bridges ($T \rightsquigarrow T'$) with the ends at points $\{A^{[T]}, B^{[T]}\}$ and $\{A^{[T']}, B^{[T']}\}$ respectively.

We turn to the next step. Here we introduce a new “parameter”, a non-negative L_p -function $h : \mathbf{R}^2 \rightarrow \mathbf{R}_+$. We fix a constant $q \in (2, p)$ (for instance we may take $q = (p+2)/2$) and again consider a mapping $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ satisfying condition (2.3) for every bridge $T \in \mathcal{BR}_E$ with ends at points $A^{[T]}, B^{[T]} \in E$. We assume that g also satisfies the following condition: for every pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \rightsquigarrow T'$,

$$(2.4) \quad \|g(T) - g(T')\| \leq \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h^q(z) dz \right)^{\frac{1}{q}}.$$

where $Q(T, T') = Q(A^{[T]}, D(T, T'))$. Here $\gamma \geq 1$ is an absolute constant. We prove (see Proposition 7.6) that the existence of a mapping g and a function h satisfying all the above conditions implies the existence of an extension $F \in L_p^2(\mathbf{R}^2)$ of the function f such that

$$(2.5) \quad \|F\|_{L_p^2(\mathbf{R}^2)} \leq C(p) \|h\|_{L_p(\mathbf{R}^2)}.$$

This is the last result of Section 7. We begin **Section 8** by introducing the notions of *set-valued mappings* and their *Sobolev-type selections*. More specifically, motivated by the equality (2.3), we introduce a set-valued mapping G_f which to every bridge $T \in \mathcal{BR}_E$ assigns the straight line consisting of all points $z \in \mathbf{R}^2$ which satisfy

$$\langle z, A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]}).$$

We say that a (vector valued) mapping $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ is a *selection of the set-valued mapping G_f* if $g(T) \in G_f(T)$ for every bridge $T \in \mathcal{BR}_E$. So any mapping g which satisfies (2.3) is a selection of G_f .

We observe that the inequality (2.4) resembles the classical Sobolev-Poincaré inequality (which is recalled in Proposition 3.2). This motivates us to introduce the terminology *Sobolev-type selection of the set-valued mapping G_f with respect to h* to designate any mapping $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ which satisfies conditions (2.3) and (2.4) for some non-negative function $h \in L_p(\mathbf{R}^2)$.

The remaining part of Section 8, which we shall presently describe in some detail, is a step by step construction of a non-negative function $h \in L_p(\mathbf{R}^2)$ and a Sobolev-type selection g with respect to h of G_f for which

$$(2.6) \quad \|h\|_{L_p(\mathbf{R}^2)} \leq C(p) \lambda^{\frac{1}{p}}.$$

Here λ and f are related as already mentioned above. (See (2.2).) This construction finally achieves the goal mentioned just before (2.2), since of course (2.6) and (2.5) together imply (2.2).

Many of the steps in Section 8 generalize ideas and methods for the case $p = q = \infty$ presented and used in [36] to the case of Sobolev-type selections, i.e., to the case $p < \infty$.

Remark 2.1 Note that for $p = q = \infty$ the problem of the existence of a Sobolev-type selection is a special case of the following general *Lipschitz selection problem*:

Let G be a set-valued mapping which to every point of a metric space (\mathcal{M}, ρ) assigns a convex closed subset of \mathbf{R}^n .

How do we know whether there exists a selection $g : \mathcal{M} \rightarrow \mathbf{R}^n$ of the mapping G satisfying the Lipschitz condition with respect to the metric ρ ? What is the order of magnitude of the minimal Lipschitz constant of such a selection of G ?

These and other problems related to Lipschitz selections of set-valued mapping have been studied in the author's papers [36, 35, 37]. A solution to one them given in [36], Theorem 3.14, leads us to the trace criterion presented in Remark 1.7. \triangleleft

We can now begin our description of the construction of the selection g and the function h mentioned above. It can be divided into two main parts. In the first and rather longer

part (a process of “pre-selection”) we obtain the required function h and a mapping \tilde{g} such that $\tilde{g}(T)$ is in some sense “close” to the straight line $G_f(T)$ for every bridge $T \in \mathcal{BR}_E$. In the second part a relatively simple procedure enables us to obtain g from \tilde{g} .

We begin the first part by associating a triangle $\Delta(T, T')$ to each pair of connected bridges $T, T' \in \mathcal{BR}_E$, $(T \rightsquigarrow T')$. The vertices of $\Delta(T, T')$ are the ends of the bridges T and T' . It is relevant to recall here that in Section 6, Proposition 6.6, we construct a family \mathcal{K}_E of pairwise disjoint squares

$$\{K(T, T') : T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'\}$$

with the following properties:

- (i) for each pair of connected bridges $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$, the size of the triangle $\Delta(T, T')$ is equivalent to the size of the square $K(T, T')$;
- (ii) $\Delta(T, T') \subset \gamma K(T, T')$ where $\gamma > 0$ is an absolute constant.

Then we introduce two basic elements of our construction. The first of them is a mapping $\mathcal{T}(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ which we define as follows:

At the center of every square $K(T, T') \in \mathcal{K}_E$ this mapping takes the value of *the gradient* $\nabla P_{\Delta(T, T')}[f]$ of the affine polynomial $P_{\Delta(T, T')}[f]$ which interpolates the function f at the vertices of the triangle $\Delta(T, T')$. At all other points we set $\mathcal{T}(f) \equiv 0$.

Note that the vector $\nabla P_{\Delta(T, T')}[f] \in \mathbf{R}^2$ has a simple geometrical description: *it coincides with the point of intersection of the straight lines $G_f(T)$ and $G_f(T')$.*

The second element is a discrete Borel measure on \mathbf{R}^2 which we denote by μ_E . This measure is supported on the set of centers of squares of the family \mathcal{K}_E . For every square $K = K(T, T') \in \mathcal{K}_E$ the μ_E -measure of its center equals $\mathbf{c}_\Delta |K|$ where \mathbf{c}_Δ is the Menger curvatures of the triangle $\Delta = \Delta(T, T')$. (See (1.1).)

We prove that the mapping $\mathcal{T}(f)$ belongs to the space $\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$ and that its norm in this space satisfies $\|\mathcal{T}(f)\|_{\vec{\Sigma}} \leq C\lambda^{\frac{1}{p}}$. The proof of this fact uses a description of the norm in the space $\vec{\Sigma}$ given in [40]. This description enables us to decompose $\mathcal{T}(f)$ into the sum $\mathcal{T}(f) = \mathcal{T}_1(f) + \mathcal{T}_2(f)$ of a mapping $\mathcal{T}_1(f) \in \vec{L}_p^1(\mathbf{R}^2)$ and a mapping $\mathcal{T}_2(f) \in \vec{L}_p(\mathbf{R}^2; \mu_E)$ provided the constant λ is related to the function f by the conditions (i), (ii) of Theorem 1.2. The norms of these mappings in the corresponding spaces are bounded by a constant $C(p)\lambda^{\frac{1}{p}}$.

We can next introduce a mapping $\tilde{g}(f) : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ which, to every bridge $T \in \mathcal{BR}_E$ whose ends are the points $A^{[T]}, B^{[T]} \in E$, assigns the vector

$$\tilde{g}(T; f) = \mathcal{T}_1(A^{[T]}; f).$$

We show that the mapping $\tilde{g}(f)$ satisfies the condition (2.4) when we choose $h_f = \|\nabla \mathcal{T}_1(f)\|$. Since $\mathcal{T}_1(f) \in \vec{L}_p^1(\mathbf{R}^2)$ and $\|\mathcal{T}_1(f)\|_{\vec{L}_p^1(\mathbf{R}^2)} \leq C\lambda^{\frac{1}{p}}$, the function h_f belongs to the space $L_p(\mathbf{R}^2)$ and $\|h_f\|_{L_p(\mathbf{R}^2)} \leq C\lambda^{\frac{1}{p}}$. Thus $\tilde{g}(f)$ is a Sobolev-type mapping with respect to h_f .

However we cannot guarantee that $\tilde{g}(T; f) \in G_f(T)$ for every bridge $T \in \mathcal{BR}_E$. This means that *in general $\tilde{g}(f)$ is not a selection of G_f .* We refer to the mapping $\tilde{g}(f)$ as a *Sobolev-type pre-selection* of the set-valued mapping G_f (with respect to the function h_f).

It turns out that the pre-selection $\tilde{g}(f)$ possesses various additional properties which will enable us to transform $\tilde{g}(f)$ into a true selection g of G_f satisfying the Sobolev-type condition (2.4) with respect to a certain function $h \in L_p(\mathbf{R}^2)$. In particular, we know that the $L_p(\mathbf{R}^2; \mu_E)$ -norm of the component $\mathcal{T}_2(f)$ is at most $C\lambda^{\frac{1}{p}}$. Using this property we show that, for each bridge $T \in \mathcal{BR}_E$, in spite of the fact that the pre-selection $\tilde{g}(T; f)$ does not necessarily lie on the straight line $G_f(T)$, it lies “rather close” to $G_f(T)$.

This concludes the first part of our construction. We turn to the second and final part: Given a bridge $T \in \mathcal{BR}_E$ we define the required selection $g(T; f)$ as *the orthogonal projection of the pre-selection $\tilde{g}(T; f)$ onto the straight line $G_f(T)$* .

Using the “closeness” of the pre-selection $\tilde{g}(T; f)$ to $G_f(T)$ we prove that the mapping $g(f) : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ is a Sobolev-type selection of the set-valued mapping G_f with respect to a certain non-negative function $h \in L_p(\mathbf{R}^2)$ whose norm satisfies $\|h\|_{L_p(\mathbf{R}^2)} \leq C\lambda^{\frac{1}{p}}$. Hence, by (2.5), there exists a function $F \in L_p^2(\mathbf{R}^2)$ such that $F|_E = f$ and

$$\|F\|_{L_p^2(\mathbf{R}^2)} \leq C(p)\|h\|_{L_p(\mathbf{R}^2)} \leq C(p)\lambda^{\frac{1}{p}}.$$

This completes the proof of the Main Theorem 1.2.

Section 9 further discusses some aspects of one of the main ingredients of the proof of Theorem 1.2, namely an almost optimal decomposition of the mapping $\mathcal{T}(f)$ into the sum $\mathcal{T}(f) = \mathcal{T}_1(f) + \mathcal{T}_2(f)$ of a mapping $\mathcal{T}_1(f) \in \vec{L}_p^1(\mathbf{R}^2)$ and a mapping $\mathcal{T}_2(f) \in \vec{L}_p(\mathbf{R}^2; \mu_E)$. As already mentioned above, criteria for the existence of such a decomposition are established in [40]. Here, using methods of [40] we describe a constructive algorithm which provides such a decomposition.

In **Section 10** we re-examine all the methods and ideas which we have used in the proof of Theorem 1.2 and present a constructive algorithm which, to every function f defined on a finite set E , assigns its almost optimal extension to a function $F(f) \in L_p^2(\mathbf{R}^2)$. At every step of this algorithm we verify that its corresponding components which we construct at this step depend on f linearly. Consequently, at the final step we obtain an almost optimal extension of f which *depends linearly on f* . This gives a constructive proof of Theorem 1.12, i.e., the existence of a continuous linear extension operator $Ext_E : L_p^2(\mathbf{R}^2)|_E \rightarrow L_p^2(\mathbf{R}^2)$ whose operator norm satisfies $\|Ext_E\| \leq C(p)$.

In **Section 11** we prove Theorems 1.4, 1.6 and 1.11. The proof of Theorem 1.4 is a slight modification of the proof of Theorem 1.2. More specifically, we replace in this proof the criterion of the norm in the space $\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$ given in Theorem 8.3 with its refinement presented in Theorem 11.1.

We prove Theorem 1.6 using another refinement of Theorem 8.3 given in Theorem 11.2. We obtain a new criterion for the norm of a function in the trace space $L_p^2(\mathbf{R}^2)|_E$ which we present in Theorem 11.3. We “sparse” this new criterion using the p -sparsification Theorem 11.4 proven by B. Klartag. (In turn this result uses a sparsification theorem by J. D. Batson, D. A. Spielman and N. Srivastava [1] as a black box.) This leads us to a proof of Theorem 1.6.

3. Main Theorem 1.2: Necessity.

Let us fix several additional notation. Throughout the paper $\gamma, \gamma_1, \gamma_2, \dots$, and C, C_1, C_2, \dots will be generic positive constants which depend only on parameters determining function spaces (p, q , etc). These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(p)$. We write $A \sim B$ if there is a constant $C \geq 1$ such that $A/C \leq B \leq CA$.

We let $Q(c, r)$ denote the square in \mathbf{R}^2 centered at c with side length $2r$. Given a square Q by c_Q we denote its center and by r_Q half of its side length. Thus $Q = Q(c_Q, r_Q)$ and $\lambda Q = Q(c_Q, \lambda r_Q)$ for every constant $\lambda > 0$.

Given $x = (x_1, x_2) \in \mathbf{R}^2$ by $\|x\| := \max\{|x_1|, |x_2|\}$ and by $\|x\|_2 := (|x_1|^2 + |x_2|^2)^{\frac{1}{2}}$ we denote the uniform and the Euclidean norms in \mathbf{R}^2 respectively.

Let $A, B \subset \mathbf{R}^2$. We put $\text{diam } A := \sup\{\|a - a'\| : a, a' \in A\}$ and

$$\text{dist}(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}.$$

For $x \in \mathbf{R}^2$ we also set $\text{dist}(x, A) := \text{dist}(\{x\}, A)$. We put $\text{dist}(A, B) = +\infty$ and $\text{dist}(x, B) = +\infty$ whenever $B = \emptyset$.

By $\langle \cdot, \cdot \rangle$ we denote the standard inner product in \mathbf{R}^2 . Given a finite set A by $\#A$ we denote the number of its elements. Finally, we let \mathcal{P}_1 denote the space of all polynomials of degree at most 1 on \mathbf{R}^2 .

In this section we prove that

$$\inf \lambda^{\frac{1}{p}} \leq C(p) \|f\|_{L_p^2(\mathbf{R}^2)|_E}$$

where λ is the parameter from Theorem 1.2 and E is an *arbitrary* (not necessarily finite) closed subset of \mathbf{R}^2 . This inequality follows from the next

Statement 3.1 *Let E be a closed subset of \mathbf{R}^2 and let $F \in L_p^2(\mathbf{R}^2)$ be a C^1 -function on \mathbf{R}^2 such that $F|_E = f$. Then the conditions (i) and (ii) of Theorem 1.2 are satisfied with $\lambda = C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}^p$ and arbitrary $\gamma \in [1, \infty)$.*

3.1. Sobolev-Poincaré type inequalities for interpolating polynomials. Our proof of Statement 3.1 uses Proposition 3.2 which presents the classical Sobolev imbedding inequality for the space $L_p^2(\mathbf{R}^2)$ whenever $p > 2$, see, e.g. [27], p. 61, or [28], p. 55. (This inequality is also known in the literature as Sobolev-Poincaré inequality for Sobolev L_p^2 -functions.)

Proposition 3.2 *Let $F \in L_p^2(\mathbf{R}^2)$ be a C^1 -function on \mathbf{R}^2 and let $2 < q \leq p < \infty$. Then for every square $Q \subset \mathbf{R}^2$ and every $x, y \in Q$ the following inequalities*

$$(3.1) \quad |F(x) - F(y) - \langle \nabla F(y), x - y \rangle| \leq C(q) \|x - y\| \text{diam } Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}},$$

$$(3.2) \quad \|\nabla F(x) - \nabla F(y)\| \leq C(q) \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}},$$

hold.

Remark 3.3 Note that inequalities (3.1) and (3.2) are equivalent to the inequality

$$\max_{Q_{xy}} |T_x[F] - T_y[F]| \leq C(q) \|x - y\|^2 \left(\frac{1}{|Q_{xy}|} \int_{Q_{xy}} (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

Here given $a \in \mathbf{R}^2$

$$T_a[F](z) = F(a) + \langle \nabla F(a), z - a \rangle$$

is the first order Taylor polynomial of the function $F \in C^1(\mathbf{R}^2) \cap L_p^2(\mathbf{R}^2)$ at the point a . By Q_{xy} we denote a square $Q(x, \|x - y\|)$. \triangleleft

First let us prove that condition (i) of Theorem 1.2 holds with $\lambda = C(p) \|F\|_{L_p^2(\mathbf{R}^2)}^p$ provided $F \in L_p^2(\mathbf{R}^2) \cap C^1(\mathbf{R}^2)$ and $F|_E = f$.

Lemma 3.4 *Let $F \in L_p^2(\mathbf{R}^2)$ be a C^1 -function on \mathbf{R}^2 , and let $2 < q \leq p < \infty$. Let Q be a square in \mathbf{R}^2 which contains three collinear points $z_1, z_2, z_3 \in \mathbf{R}^2$ such that $z_2 \in (z_1, z_3)$. Then*

$$\left| \frac{F(z_1) - F(z_2)}{\|z_1 - z_2\|_2} - \frac{F(z_2) - F(z_3)}{\|z_2 - z_3\|_2} \right| \leq C(q) \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

Proof. Since F is a C^1 -function, there exist points $z', z'' \in (z_1, z_3)$ such that

$$F(z_1) - F(z_2) = \langle \nabla F(z'), z_1 - z_2 \rangle \quad \text{and} \quad F(z_2) - F(z_3) = \langle \nabla F(z''), z_2 - z_3 \rangle.$$

Hence

$$\begin{aligned} I &:= \left| \frac{F(z_1) - F(z_2)}{\|z_1 - z_2\|_2} - \frac{F(z_2) - F(z_3)}{\|z_2 - z_3\|_2} \right| \\ &= \left| \left\langle \nabla F(z'), \frac{z_1 - z_2}{\|z_1 - z_2\|_2} \right\rangle - \left\langle \nabla F(z''), \frac{z_2 - z_3}{\|z_2 - z_3\|_2} \right\rangle \right|. \end{aligned}$$

Since $z_2 \in (z_1, z_3)$,

$$\frac{z_1 - z_2}{\|z_1 - z_2\|_2} = \frac{z_2 - z_3}{\|z_2 - z_3\|_2} = \vec{n}$$

where $\vec{n} \in \mathbf{R}^2$ is a vector with $\|\vec{n}\|_2 = 1$. Hence, by (3.2),

$$\begin{aligned} I &= |\langle \nabla F(z') - \nabla F(z''), \vec{n} \rangle| \\ &\leq \|\nabla F(z') - \nabla F(z'')\|_2 \leq C(q) \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \end{aligned}$$

proving the lemma. ■

Let $\gamma \geq 1$ and let $q := (p+2)/2$. Clearly $2 < q < p$. Let Q be a square in \mathbf{R}^2 and let $Z = \{z_1, z_2, z_3\}$ be a subset of E such that $Z \subset \gamma Q$ and $z_2 \in (z_1, z_3)$. Since $F|_E = f$, by Lemma 3.4,

$$I(Z; f) := \left| \frac{f(z_1) - f(z_2)}{\|z_1 - z_2\|_2} - \frac{f(z_2) - f(z_3)}{\|z_2 - z_3\|_2} \right| \leq C \operatorname{diam}(\gamma Q) \left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{1}{q}}$$

so that

$$(3.3) \quad I(Z; f)^p (\operatorname{diam} Q)^{2-p} \leq C |Q| \left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{p}{q}}$$

where $C = C(p, \gamma)$.

Let $y \in Q$. As usual given a function $g \in L_{1,loc}(\mathbf{R}^2)$ by $\mathcal{M}[g](y)$ we denote the Hardy-Littlewood maximal function

$$\mathcal{M}[g](y) := \sup_{K \ni y} \frac{1}{|K|} \int_K |g(z)| dz.$$

Here the supremum is taken over all squares K containing y . Hence

$$\left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \leq (\mathcal{M}[(\nabla^2 F)^q](y))^{\frac{p}{q}}.$$

Integrating this inequality over square Q we obtain

$$(3.4) \quad |Q| \left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \leq \int_Q \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz.$$

Combining this inequality with (3.3), we obtain

$$(3.5) \quad I(Z; f)^p (\operatorname{diam} Q)^{2-p} \leq C \int_Q \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz.$$

Let $\{Q_i : i = 1, \dots, m\}$ be a finite family of pairwise disjoint squares. Consider three arbitrary collinear points $z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \in E \cap (\gamma Q_i)$ such that $z_2^{(i)} \in (z_1^{(i)}, z_3^{(i)})$, $i = 1, \dots, m$. We have

$$J := \sum_{i=1}^m \left| \frac{f(z_1^{(i)}) - f(z_2^{(i)})}{\|z_1^{(i)} - z_2^{(i)}\|} - \frac{f(z_2^{(i)}) - f(z_3^{(i)})}{\|z_2^{(i)} - z_3^{(i)}\|} \right|^p (\operatorname{diam} Q_i)^{2-p} = \sum_{i=1}^m I(Z_i; f)^p (\operatorname{diam} Q_i)^{2-p}$$

so that, by (3.5),

$$J \leq C \sum_{i=1}^m \int_{Q_i} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz \leq C \int_{\mathbf{R}^2} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz$$

where $C = C(p, \gamma)$. Since $p/q > 1$, by the Hardy-Littlewood maximal theorem,

$$J \leq C \int_{\mathbf{R}^2} ((\nabla^2 F)^q)^{\frac{p}{q}} dz = C \int_{\mathbf{R}^2} (\nabla^2 F)^p dz = C \|F\|_{L_p^2(\mathbf{R}^2)}^p.$$

This inequality proves that for every function $F \in L_p^2(\mathbf{R}^2) \cap C^1(\mathbf{R}^2)$ such that $F|_E = f$ the condition (i) of Theorem 1.2 holds with the constant $\lambda = C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}^p$.

Now let us prove that for every $\gamma \in [1, \infty)$ and every function $F \in L_p^2(\mathbf{R}^2) \cap C^1(\mathbf{R}^2)$ such that $F|_E = f$ the condition (ii) of Theorem 1.2 holds with $\lambda = C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}^p$.

We will be needed two auxiliary results. The first of them is the following

Proposition 3.5 *Let $F \in L_p^2(\mathbf{R}^2)$ be a C^1 -function on \mathbf{R}^2 , and let $2 < q \leq p < \infty$. Let $\Delta = \{z_1, z_2, z_3\}$ be a triangle. Then for every square $Q \supset \Delta$ and every $x \in Q$ we have*

$$(3.6) \quad \|\nabla P_\Delta[F] - \nabla F(x)\| \leq C(q) \frac{R_\Delta}{\text{diam } \Delta} \text{diam } Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

Recall that R_Δ denotes the radius of the circle passing through the points z_1, z_2, z_3 , and $P_\Delta[F]$ denotes the affine polynomial interpolating F at the points z_1, z_2 and z_3 .

Proof. Without loss of generality we may assume that

$$(3.7) \quad \|z_1 - z_2\|_2 \leq \|z_2 - z_3\|_2 \leq \|z_1 - z_3\|_2.$$

Note that, by (3.2),

$$\|\nabla F(x) - \nabla F(z_1)\| \leq C(q) \text{diam } Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}},$$

so that it suffice to prove the proposition for the case $x = z_1$.

Note that $P_\Delta[F] \in \mathcal{P}_1$ so that

$$P_\Delta[F](u) = P_\Delta[F](z_1) + \langle \nabla P_\Delta[F], u - z_1 \rangle, \quad u \in \mathbf{R}^2.$$

Also recall that $P_\Delta[F]$ interpolates F on the vertices of Δ so that

$$P_\Delta[F](z_i) = F(z_i), \quad i = 1, 2, 3.$$

Let

$$T_{z_1}[F](u) = F(z_1) + \langle \nabla F(z_1), u - z_1 \rangle = P_\Delta[F](z_1) + \langle \nabla F(z_1), u - z_1 \rangle, \quad u \in \mathbf{R}^2,$$

be the first order Taylor polynomial of F at the point z_1 . Then, by (3.1),

$$|P_\Delta[F](z_2) - T_{z_1}[F](z_2)| = |F(z_2) - T_{z_1}[F](z_2)| \leq C \|z_1 - z_2\|_2 \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

On the hand,

$$P_\Delta[F](z_2) - T_{z_1}[F](z_2) = \langle \nabla P_\Delta[F] - \nabla F(z_1), z_2 - z_1 \rangle$$

so that

$$(3.8) \quad \left| \left\langle \nabla P_\Delta[F] - \nabla F(z_1), \frac{z_2 - z_1}{\|z_2 - z_1\|_2} \right\rangle \right| \leq C \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

In a similar way we prove that

$$(3.9) \quad \left| \left\langle \nabla P_\Delta[F] - \nabla F(z_1), \frac{z_3 - z_1}{\|z_3 - z_1\|_2} \right\rangle \right| \leq C \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

Now let

$$\vec{a} := \nabla P_\Delta[F] - \nabla F(z_1), \quad \vec{n} := \frac{z_2 - z_1}{\|z_2 - z_1\|_2}, \quad \vec{m} := \frac{z_3 - z_1}{\|z_3 - z_1\|_2},$$

and

$$A := C \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

Then, by (3.8) and (3.9),

$$(3.10) \quad |\langle \vec{a}, \vec{n} \rangle| \leq A \quad \text{and} \quad |\langle \vec{a}, \vec{m} \rangle| \leq A.$$

Let $\alpha \in (0, \pi)$ be the angle between the sides $[z_1, z_2]$ and $[z_1, z_3]$ of the triangle Δ . Thus α is also the angle between \vec{n} and \vec{m} .

Let \vec{n}_1 be a unit vector in \mathbf{R}^2 which is orthogonal to \vec{n} , i.e., $\|\vec{n}_1\|_2 = 1$ and $\langle \vec{n}, \vec{n}_1 \rangle = 0$. Then $\vec{m} = (\cos \alpha) \vec{n} + \varepsilon (\sin \alpha) \vec{n}_1$ where $\varepsilon \in \{1, -1\}$. Hence

$$|\langle \vec{a}, \vec{n}_1 \rangle| = \frac{1}{\sin \alpha} |\langle \vec{a}, \vec{m} - (\cos \alpha) \vec{n} \rangle| \leq \frac{1}{\sin \alpha} (|\langle \vec{a}, \vec{m} \rangle| + |\langle \vec{a}, \vec{n} \rangle|)$$

so that, by (3.10), $|\langle \vec{a}, \vec{n}_1 \rangle| \leq 2A / \sin \alpha$. We obtain

$$\|a\|_2 = (\langle \vec{a}, \vec{n} \rangle^2 + \langle \vec{a}, \vec{n}_1 \rangle^2)^{\frac{1}{2}} \leq \left(A^2 + \left(\frac{2A}{\sin \alpha} \right)^2 \right)^{\frac{1}{2}} \leq C_1 A / \sin \alpha.$$

Hence

$$(3.11) \quad \|\nabla P_\Delta[F] - \nabla F(z_1)\| \leq C \frac{\operatorname{diam} Q}{\sin \alpha} \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

Recall that α is the angle in the triangle Δ which is opposite to the side $[z_2, z_3]$ so that

$$R_\Delta = \frac{\|z_2 - z_3\|_2}{2 \sin \alpha},$$

see, e.g. [31], p. 29. But, by (3.7) and the triangle inequality,

$$\text{diam } \Delta = \|z_1 - z_3\|_2 \leq 2\|z_2 - z_3\|_2$$

so that

$$\frac{1}{\sin \alpha} = \frac{2R_\Delta}{\|z_2 - z_3\|_2} \leq \frac{4R_\Delta}{\text{diam } \Delta}.$$

Combining this inequality with (3.11) we obtain the required inequality (3.6). ■

Proposition 3.5 and inequality (3.1) imply the following

Proposition 3.6 *Let $2 < q \leq p < \infty$ and let $\Delta = \{z_1, z_2, z_3\}, \Delta' = \{z'_1, z'_2, z'_3\}$ be two triangles in \mathbf{R}^2 . Let Q, Q', \tilde{Q} be squares in \mathbf{R}^2 such that*

$$\Delta \subset Q, \quad \Delta' \subset Q', \quad Q \cup Q' \subset \tilde{Q}.$$

Then for every C^1 -function $F \in L_p^2(\mathbf{R}^2)$ the following inequality

$$\begin{aligned} \|\nabla P_\Delta[F] - \nabla P_{\Delta'}[F]\| &\leq C(q) \left\{ \frac{R_\Delta}{\text{diam } \Delta} \text{diam } Q \left(\frac{1}{|Q|} \int_Q (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \right. \\ &+ \left. \frac{R_{\Delta'}}{\text{diam } \Delta'} \text{diam } Q' \left(\frac{1}{|Q'|} \int_{Q'} (\nabla^2 F)^q dz \right)^{\frac{1}{q}} + \text{diam } \tilde{Q} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \right\} \end{aligned}$$

holds.

3.2. Part (ii) of the necessity and the space $\vec{\Sigma} := \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$. The next auxiliary result relates to an optimal decomposition of a function into a sum of a Sobolev function and an L_p -weighted function.

Let μ be a non-trivial non-negative Borel measure on \mathbf{R}^2 . Let $2 < p < \infty$ and let $L_p(\mathbf{R}^2; \mu)$ be the L_p -space on \mathbf{R}^2 with respect to the measure μ . We norm this space by

$$\|f\|_{L_p(\mathbf{R}^2; \mu)} = \left(\int_{\mathbf{R}^2} |f|^p d\mu \right)^{\frac{1}{p}}.$$

By Σ we denote the space $L_p^1(\mathbf{R}^2) + L_p(\mathbf{R}^2; \mu)$ equipped with the norm:

$$\|f\|_\Sigma = \inf \{ \|f_1\|_{L_p^1(\mathbf{R}^2)} + \|f_2\|_{L_p(\mathbf{R}^2; \mu)} : f = f_1 + f_2, f_1 \in L_p^1(\mathbf{R}^2), f_2 \in L_p(\mathbf{R}^2; \mu) \}.$$

Let us also define vector versions of the above spaces. We let $\vec{L}_p^1(\mathbf{R}^2)$ denote the space of Sobolev mappings $\vec{F} = (F_1, F_2) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ whose components $F_1, F_2 \in L_p^1(\mathbf{R}^2)$. This space is normed by

$$(3.12) \quad \|\vec{F}\|_{\vec{L}_p^1(\mathbf{R}^2)} := \|F_1\|_{L_p^1(\mathbf{R}^2)} + \|F_2\|_{L_p^1(\mathbf{R}^2)}.$$

In turn, by $\vec{L}_p(\mathbf{R}^2; \mu)$ we denote the space of all mappings $\vec{F} = (F_1, F_2) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ whose components $F_1, F_2 \in L_p(\mathbf{R}^2; \mu)$. We norm $\vec{L}_p(\mathbf{R}^2; \mu)$ by

$$\|\vec{F}\|_{\vec{L}_p(\mathbf{R}^2; \mu)} := \|F_1\|_{L_p(\mathbf{R}^2; \mu)} + \|F_2\|_{L_p(\mathbf{R}^2; \mu)}.$$

By $\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$ we denote the sum of these spaces. The space $\vec{\Sigma}$ is normed by

$$\|\vec{F}\|_{\vec{\Sigma}} = \inf\{\|\vec{F}_1\|_{\vec{L}_p^1(\mathbf{R}^2)} + \|\vec{F}_2\|_{\vec{L}_p(\mathbf{R}^2; \mu)} : \vec{F} = \vec{F}_1 + \vec{F}_2, \vec{F}_1 \in \vec{L}_p^1(\mathbf{R}^2), \vec{F}_2 \in \vec{L}_p(\mathbf{R}^2; \mu)\}.$$

In [40], Theorem 2.4, we present a necessary condition for a function to belong to the space $\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$ whenever $p > 2$. Applying this result to every component of a mapping $\mathcal{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ we obtain the following

Theorem 3.7 *Let $2 < p < \infty, 1 \leq \gamma < \infty$, and let μ be a non-trivial non-negative Borel measure on \mathbf{R}^2 . Suppose that a mapping $\mathcal{T} \in \vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$. Then the following statement is true:*

Let \mathcal{A} be a finite family of squares in \mathbf{R}^2 , and let \mathcal{S} be a finite family of closed subsets in \mathbf{R}^2 with covering multiplicity $M(\mathcal{A}), M(\mathcal{S}) \leq N$. Suppose that to every square $Q \in \mathcal{A}$ we have assigned two subsets $S'_Q, S''_Q \in \mathcal{S}$ such that

$$(3.13) \quad S'_Q \cup S''_Q \subset \gamma Q.$$

Then

$$(3.14) \quad \sum_{Q \in \mathcal{A}} \frac{(\text{diam } Q)^{2-p} \iint_{S'_Q \times S''_Q} \|\mathcal{T}(x) - \mathcal{T}(y)\|^p d\mu(x) d\mu(y)}{\{(\text{diam } S'_Q)^{2-p} + \mu(S'_Q)\} \{(\text{diam } S''_Q)^{2-p} + \mu(S''_Q)\}} \leq C(\gamma, N) \|\mathcal{T}\|_{\vec{\Sigma}}^p.$$

We are in a position to prove the condition (ii) of Theorem 1.2 with $\lambda = C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}^p$ for every $\gamma \in [1, \infty)$ and every C^1 -function $F \in L_p^2(\mathbf{R}^2)$ such that $F|_E = f$.

Let \mathcal{K} be a finite family of pairwise disjoint squares. Given $K \in \mathcal{K}$ let $\Delta(K)$ be a triangle satisfying conditions (1.4) with some constant $\gamma \in [1, \infty)$. Let

$$\mathcal{C} := \{c_K : K \in \mathcal{K}\}$$

be the family of centers of all squares from \mathcal{K} .

We introduce a discrete Borel measure μ on \mathbf{R}^2 with $\text{supp } \mu := \mathcal{C}$ as follows: for every square $K \in \mathcal{K}$ we put

$$(3.15) \quad \mu(\{c_K\}) := \mathbf{c}_{\Delta(K)}^p |K|.$$

Thus for every set $S \subset \mathbf{R}^2$ we have

$$\mu(S) = \sum \left\{ \mathbf{c}_{\Delta(K)}^p |K| : K \in \mathcal{K}, c_K \in S \right\}.$$

Proposition 3.8 *Let $p \in (2, \infty)$ and let $F \in L_p^2(\mathbf{R}^2)$. Let $\mathcal{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a mapping such that*

$$(3.16) \quad \mathcal{T}(c_K) := \nabla P_{\Delta(K)}[F] \quad \text{for every } K \in \mathcal{K},$$

and

$$(3.17) \quad \mathcal{T}(x) := 0 \quad \text{whenever } x \in \mathbf{R}^2 \setminus \mathcal{C}.$$

Then $\mathcal{T} \in \vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$. Furthermore,

$$(3.18) \quad \|\mathcal{T}\|_{\vec{\Sigma}} \leq C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}.$$

Proof. Fix $q \in (2, p)$ (say $q = (p + 2)/2$) and a square $K \in \mathcal{K}$. Applying Proposition 3.5 to the square $Q = \gamma K$ and arbitrary $x = c_K$ we obtain

$$\|\nabla P_{\Delta(K)}[F] - \nabla F(c_K)\| \leq CR_{\Delta(K)} \frac{\text{diam } K}{\text{diam } \Delta(K)} \left(\frac{1}{|\gamma K|} \int_{\gamma K} (\nabla^2 F)^q dz \right)^{\frac{1}{q}}.$$

where $C = C(p, \gamma)$. Since $\text{diam } \Delta(K) \sim \text{diam } K$, we have

$$\begin{aligned} \|\nabla P_{\Delta(K)}[F] - \nabla F(c_K)\| &\leq CR_{\Delta(K)} \left(\frac{1}{|\gamma K|} \int_{\gamma K} (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \\ &\leq CR_{\Delta(K)} \mathcal{M}[(\nabla^2 F)^q]^{\frac{1}{q}}(y), \quad y \in K. \end{aligned}$$

(Recall that \mathcal{M} denotes the Hardy-Littlewood maximal function.) Hence

$$\|\nabla P_{\Delta(K)}[F] - \nabla F(c_K)\|^p \mathbf{c}_{\Delta(K)}^p \leq C \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}}(y), \quad y \in K.$$

Integrating this inequality over square K we obtain

$$\|\nabla P_{\Delta(K)}[F] - \nabla F(x)\|^p \mathbf{c}_{\Delta(K)}^p |K| \leq C \int_K \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}}(y) dy.$$

Hence, by (3.16) and (3.17),

$$\begin{aligned} \int_{\mathbf{R}^2} \|\mathcal{T}(x) - \nabla F(x)\|^p d\mu(x) &= \sum_{K \in \mathcal{K}} \|\nabla P_{\Delta(K)}[F] - \nabla F(x)\|^p \mathbf{c}_{\Delta(K)}^p |K| \\ &\leq C \sum_{K \in \mathcal{K}} \int_K \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}}(y) dy \\ &\leq C \int_{\mathbf{R}^2} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}}(y) dy. \end{aligned}$$

Since $p/q > 1$, by the Hardy-Littlewood maximal theorem,

$$\int_{\mathbf{R}^2} \|\mathcal{T}(x) - \nabla F(c_K)\|^p d\mu(x) \leq C \int_{\mathbf{R}^2} ((\nabla^2 F)^q)^{\frac{p}{q}}(y) dy = C \int_{\mathbf{R}^2} (\nabla^2 F)^p(y) dy$$

proving that

$$\|\mathcal{T} - \nabla F\|_{\tilde{L}_p(\mathbf{R}^2; \mu)}^p = \int_{\mathbf{R}^2} \|\mathcal{T}(x) - \nabla F(x)\|^p d\mu(x) \leq C \|F\|_{L_p^2(\mathbf{R}^2)}^p.$$

Finally,

$$\|\mathcal{T}\|_{\tilde{\Sigma}} \leq \|\nabla F\|_{\tilde{L}_p^1(\mathbf{R}^2)} + \|\mathcal{T} - \nabla F\|_{\tilde{L}_p(\mathbf{R}^2; \mu)} \leq C \|F\|_{L_p^2(\mathbf{R}^2)}.$$

The proposition is proved. ■

Let \mathcal{Q} be a finite family of squares with covering multiplicity $M(\mathcal{Q}) \leq N$. Given $Q \in \mathcal{Q}$ let $Q', Q'' \in \mathcal{Q}$ be squares such that $Q' \cup Q'' \subset \gamma Q$.

Let us apply Theorem 3.7 to the mapping \mathcal{T} whenever

$$(3.19) \quad \mathcal{A} = \mathcal{S} = \mathcal{Q}$$

and $S'_Q = Q', S''_Q = Q''$ for arbitrary $Q \in \mathcal{A}$. Then inclusion (3.13) is equivalent to inclusion $Q' \cup Q'' \subset \gamma Q$. Furthermore, $\text{diam } S'_Q = \text{diam } Q', \text{diam } S''_Q = \text{diam } Q''$ and, by (1.5),

$$\mu(S'_Q) = \sigma_p(Q'; \mathcal{K}) \quad \text{and} \quad \mu(S''_Q) = \sigma_p(Q''; \mathcal{K}).$$

Finally,

$$\begin{aligned} & \iint_{S'_Q \times S''_Q} \|\mathcal{T}(x) - \mathcal{T}(y)\|^p d\mu(x) d\mu(y) \\ &= \sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in Q'}} \sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in Q''}} \|\nabla P_{\Delta(K')}[F] - \nabla P_{\Delta(K'')}[F]\|^p \mathbf{c}_{\Delta(K')}^p |K'| \mathbf{c}_{\Delta(K'')}^p |K''| \\ &= S_p(F : Q', Q''; \mathcal{K}). \end{aligned}$$

But $F|_E = f$ so that $S_p(F : Q', Q''; \mathcal{K}) = S_p(f : Q', Q''; \mathcal{K})$ proving that

$$\iint_{S'_Q \times S''_Q} \|\mathcal{T}(x) - \mathcal{T}(y)\|^p d\mu(x) d\mu(y) = S_p(f : Q', Q''; \mathcal{K}).$$

Hence, by inequalities (3.14) and (3.18),

$$\begin{aligned} & \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} S_p(f : Q', Q''; \mathcal{K})}{\{(\text{diam } Q')^{2-p} + \sigma_p(Q'; \mathcal{K})\} \{(\text{diam } Q'')^{2-p} + \sigma_p(Q''; \mathcal{K})\}} \\ &= \sum_{Q \in \mathcal{A}} \frac{(\text{diam } Q)^{2-p} \iint_{S'_Q \times S''_Q} \|\mathcal{T}(x) - \mathcal{T}(y)\|^p d\mu(x) d\mu(y)}{\{(\text{diam } S'_Q)^{2-p} + \mu(S'_Q)\} \{(\text{diam } S''_Q)^{2-p} + \mu(S''_Q)\}} \\ &\leq C \|\mathcal{T}\|_{\tilde{\Sigma}}^p \leq C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}^p. \end{aligned}$$

This proves that for every function $F \in L_p^2(\mathbf{R}^2) \cap C^1(\mathbf{R}^2)$ such that $F|_E = f$ the condition (ii) of Theorem 1.2 holds with $\lambda = C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}^p$.

3.3. A direct proof of part (ii) of the necessity. As we have mentioned in Section 2, for the reader's convenience, we also give a direct proof of part (ii) of Theorem 1.2 which does not use any results of [40].

We will prove a general result which implies the statement of part (ii).

Let $\gamma \geq 1$ be a constant. Let \mathcal{Q} be a finite families of pairwise disjoint squares and let \mathcal{S} be a finite families of pairwise disjoint sets in \mathbf{R}^2 . Given a square $Q \in \mathcal{Q}$ let S'_Q, S''_Q be a pair of sets from \mathcal{S} such that $S'_Q \cup S''_Q \subset \gamma Q$.

Let \mathcal{K} be a finite family of pairwise disjoint squares. Given $K \in \mathcal{K}$ let $\Delta(K)$ be a triangle such that

$$\Delta(K) \subset \gamma K \quad \text{and} \quad \text{diam } K \leq \gamma \text{diam } \Delta(K).$$

Let $S \in \mathcal{S}$ and let

$$(3.20) \quad \sigma_p(S; \mathcal{K}) := \sum \left\{ \mathbf{c}_{\Delta(K)}^p |K| : K \in \mathcal{K}, c_K \in S \right\}.$$

Given a function $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ and sets $S'_Q, S''_Q \in \mathcal{S}$ let

$$S_p(F : S'_Q, S''_Q; \mathcal{K}) := \sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in S'_Q}} \sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in S''_Q}} \|\nabla P_{\Delta(K')}[F] - \nabla P_{\Delta(K'')}[F]\|^p \mathbf{c}_{\Delta(K')}^p |K'| \mathbf{c}_{\Delta(K'')}^p |K''|.$$

Finally we put

$$U\{\mathcal{Q}\} := \bigcup_{Q \in \mathcal{Q}} Q \quad \text{and} \quad U\{\mathcal{K}\} := \bigcup_{K \in \mathcal{K}} K.$$

Proposition 3.9 *Let $2 < q < p$. Then for every smooth function $F \in L_p^2(\mathbf{R}^2)$ the following inequality*

$$\begin{aligned} & \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} S_p(F : S'_Q, S''_Q; \mathcal{K})}{\{(\text{diam } S'_Q)^{2-p} + \sigma_p(S'_Q; \mathcal{K})\} \{(\text{diam } S''_Q)^{2-p} + \sigma_p(S''_Q; \mathcal{K})\}} \\ & \leq C \left\{ \int_{U\{\mathcal{K}\}} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz + \int_{U\{\mathcal{Q}\}} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz \right\}. \end{aligned}$$

holds. Here C is a constant depending only on q, p , and γ .

Proof. Consider two squares $K', K'' \in \mathcal{K}$ such that $c_{K'} \in S'_Q$ and $c_{K''} \in S''_Q$. Let us apply Proposition 3.6 to squares $\gamma K', \gamma K'', \gamma Q$, and triangles $\Delta(K') \subset \gamma K'$ and $\Delta(K'') \subset \gamma K''$.

By this proposition,

$$\begin{aligned}
\|\nabla P_{\Delta(K')}[F] - \nabla P_{\Delta(K'')}[F]\| &\leq C \left\{ \frac{R_{\Delta(K')}}{\text{diam } \Delta(K')} \text{diam}(\gamma K') \left(\frac{1}{|\gamma K'|} \int_{\gamma K'} (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \right. \\
&\quad + \frac{R_{\Delta(K'')}}{\text{diam } \Delta(K'')} \text{diam}(\gamma K'') \left(\frac{1}{|\gamma K''|} \int_{\gamma K''} (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \\
&\quad \left. + \text{diam}(\gamma Q) \left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Recall that $\text{diam } \Delta(K') \sim \text{diam } K'$, $\text{diam } \Delta(K'') \sim \text{diam } K''$,. We also recall that for every triangle $\Delta \subset \mathbf{R}^2$ we have $\mathbf{c}_\Delta = 1/R_\Delta$. Hence

$$\begin{aligned}
&\|\nabla P_{\Delta(K')}[F] - \nabla P_{\Delta(K'')}[F]\|^p \mathbf{c}_{\Delta(K')}^p \mathbf{c}_{\Delta(K'')}^p |K'| |K''| \\
&\leq C \left\{ \mathbf{c}_{\Delta(K'')}^p |K''| |K'| \left(\frac{1}{|\gamma K'|} \int_{\gamma K'} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \right. \\
&\quad + \mathbf{c}_{\Delta(K')}^p |K'| |K''| \left(\frac{1}{|\gamma K''|} \int_{\gamma K''} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \\
&\quad \left. + \mathbf{c}_{\Delta(K')}^p \mathbf{c}_{\Delta(K'')}^p |K'| |K''| (\text{diam } Q)^p \left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \right\}.
\end{aligned}$$

We obtain

$$\begin{aligned}
&S_p(F : S'_Q, S''_Q; \mathcal{K}) \\
&\leq C \left\{ \left(\sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in S'_Q}} \mathbf{c}_{\Delta(K'')}^p |K''| \right) \left(\sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in S''_Q}} |K'| \left(\frac{1}{|\gamma K'|} \int_{\gamma K'} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \right) \right. \\
&\quad + \left(\sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in S'_Q}} \mathbf{c}_{\Delta(K')}^p |K'| \right) \left(\sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in S''_Q}} |K''| \left(\frac{1}{|\gamma K''|} \int_{\gamma K''} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \right) \\
&\quad \left. + \left(\sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in S'_Q}} \mathbf{c}_{\Delta(K')}^p |K'| \right) \left(\sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in S''_Q}} \mathbf{c}_{\Delta(K'')}^p |K''| \right) (\text{diam } Q)^p \left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \right\}
\end{aligned}$$

Let $S \in \mathcal{S}$ and let

$$V\{S\} := \bigcup \{K \in \mathcal{K} : c_K \in S\}.$$

For the sake of brevity we put

$$H(x) = \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}}(x), \quad x \in \mathbf{R}^2.$$

By (3.4),

$$|K'| \left(\frac{1}{|\gamma K'|} \int_{\gamma K'} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \leq \int_{K'} H(z) dz,$$

so that

$$\sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in S'_Q}} |K'| \left(\frac{1}{|\gamma K'|} \int_{\gamma K'} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \leq \sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in S'_Q}} \int_{K'} H(z) dz \leq \int_{V\{S'_Q\}} H(z) dz.$$

In the same way we prove that

$$\sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in S''_Q}} |K''| \left(\frac{1}{|\gamma K''|} \int_{\gamma K''} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \leq \int_{V\{S''_Q\}} H(z) dz.$$

Also, by (3.4),

$$|Q| \left(\frac{1}{|\gamma Q|} \int_{\gamma Q} (\nabla^2 F)^q dz \right)^{\frac{p}{q}} \leq \int_Q H(z) dz.$$

Combining these inequalities with the above estimate of $S_p(F : S'_Q, S''_Q; \mathcal{K})$ and definition (3.20), we obtain

$$\begin{aligned} S_p(F : S'_Q, S''_Q; \mathcal{K}) &\leq C \left\{ \sigma_p(S''_Q; \mathcal{K}) \int_{V\{S'_Q\}} H(z) dz + \sigma_p(S'_Q; \mathcal{K}) \int_{V\{S''_Q\}} H(z) dz \right. \\ &\quad \left. + \sigma_p(S'_Q; \mathcal{K}) \sigma_p(S''_Q; \mathcal{K}) (\text{diam } Q)^{p-2} \int_Q H(z) dz \right\} \end{aligned}$$

Let

$$A := \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} S_p(F : S'_Q, S''_Q; \mathcal{K})}{\{(\text{diam } S'_Q)^{2-p} + \sigma_p(S'_Q; \mathcal{K})\} \{(\text{diam } S''_Q)^{2-p} + \sigma_p(S''_Q; \mathcal{K})\}}.$$

Then the above estimate of $S_p(F : S'_Q, S''_Q; \mathcal{K})$ implies the following inequality:

$$(3.21) \quad A \leq C(I_1 + I_2 + I_3).$$

Here

$$I_1 := \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} \sigma_p(S''_Q; \mathcal{K})}{\{(\text{diam } S'_Q)^{2-p} + \sigma_p(S'_Q; \mathcal{K})\} \{(\text{diam } S''_Q)^{2-p} + \sigma_p(S''_Q; \mathcal{K})\}} \int_{V\{S'_Q\}} H(z) dz,$$

$$I_2 := \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} \sigma_p(S'_Q; \mathcal{K})}{\{(\text{diam } S'_Q)^{2-p} + \sigma_p(S'_Q; \mathcal{K})\} \{(\text{diam } S''_Q)^{2-p} + \sigma_p(S''_Q; \mathcal{K})\}} \int_{V\{S''_Q\}} H(z) dz,$$

and

$$I_3 := \sum_{Q \in \mathcal{Q}} \frac{\sigma_p(S'_Q; \mathcal{K}) \sigma_p(S''_Q; \mathcal{K})}{\{(\text{diam } S'_Q)^{2-p} + \sigma_p(S'_Q; \mathcal{K})\} \{(\text{diam } S''_Q)^{2-p} + \sigma_p(S''_Q; \mathcal{K})\}} \int_Q H(z) dz.$$

Clearly,

$$I_3 \leq \sum_{Q \in \mathcal{Q}} \int_Q H(z) dz.$$

Since the squares of the family \mathcal{Q} are pairwise disjoint, we obtain

$$(3.22) \quad I_3 \leq \int_{U\{\mathcal{Q}\}} H(z) dz.$$

Prove that

$$I_1 \leq \int_{U\{\mathcal{K}\}} H(z) dz.$$

Obviously,

$$I_1 \leq \sum_{Q \in \mathcal{Q}} \left(\frac{\text{diam } S'_Q}{\text{diam } Q} \right)^{p-2} \int_{V\{S'_Q\}} H(z) dz$$

so that

$$(3.23) \quad I_1 \leq \sum_{S \in \tilde{\mathcal{S}}} J(S) (\text{diam } S)^{p-2} \int_{V\{S\}} H(z) dz.$$

Here

$$\tilde{\mathcal{S}} := \{S \in \mathcal{S} : \text{diam } S > 0\}$$

and

$$J(S) := \sum \{(\text{diam } Q)^{2-p} : Q \in \mathcal{Q}, S'_Q = S\}.$$

Prove that

$$(3.24) \quad J(S) \leq C(\text{diam } S)^{2-p}$$

for every $S \in \tilde{\mathcal{S}}$. Let

$$G(S) := \{Q \in \mathcal{Q} : S'_Q = S\}.$$

Since $S \subset \gamma Q$ whenever $Q \in G(S)$, we have

$$(3.25) \quad \text{diam } S \leq \gamma \text{diam } Q \quad \text{for every } Q \in G(S).$$

Fix a point $x_0 \in S$ and introduce two family of squares:

$$G_1(S) := \{Q \in \mathcal{Q} : S'_Q = S, x_0 \in Q\},$$

and

$$G_2(S) := \{Q \in \mathcal{Q} : S'_Q = S, x_0 \notin Q\}.$$

Clearly, $G_1(S)$ and $G_2(S)$ is a partition of $G(S)$.

Since the squares of $G(S)$ are pairwise disjoint, the family $G_1(S)$ consists of at most one element. Hence, by (3.25),

$$(3.26) \quad \sum \{(\text{diam } Q)^{2-p} : Q \in G_1(S)\} \leq \gamma^{p-2} (\text{diam } S)^{2-p}.$$

Prove that

$$\sum \{(\text{diam } Q)^{2-p} : Q \in G_2(S)\} \leq C(\gamma, p) (\text{diam } S)^{2-p}.$$

In fact, let $Q \in G_2(S)$ and let $\tilde{Q} = \frac{1}{2}Q$. Since $S \subset \gamma Q$ and $\gamma \geq 1$, we have $y, x_0 \in \gamma Q$ for every $y \in \tilde{Q}$. Hence,

$$\|x_0 - y\| \leq \text{diam}(\gamma Q) = \gamma \text{diam } Q$$

so that

$$(\text{diam } Q)^{-p} \leq \gamma^p \|x_0 - y\|^{-p}, \quad y \in \tilde{Q}.$$

Integrating this inequality over \tilde{Q} , we obtain

$$(\text{diam } Q)^{2-p} \leq 4\gamma^p \int_{\tilde{Q}} \|x_0 - y\|^{-p} dy$$

so that

$$\sum \{(\text{diam } Q)^{2-p} : Q \in G_2(S)\} \leq C(\gamma, p) \int_{\tilde{Q}} \|x_0 - y\|^{-p} dy.$$

Let $U := \cup \{\tilde{Q} : Q \in G_2(S)\}$. Since the squares of the family $\{\tilde{Q} : Q \in G_2(S)\}$ are pairwise disjoint, we have

$$(3.27) \quad \sum \{(\text{diam } Q)^{2-p} : Q \in G_2(S)\} \leq C(\gamma, p) \int_U \|x_0 - y\|^{-p} dy.$$

Let $Q \in G_2(S)$ and let $y \in \tilde{Q} = \frac{1}{2}Q$. Since $x_0 \notin Q$, we have

$$\|x_0 - y\| \geq \frac{1}{4} \text{diam } Q \geq \frac{1}{4\gamma} \text{diam } S,$$

see inequality (3.25). Hence

$$(3.28) \quad U \subset \{y \in \mathbf{R}^2 : \|x_0 - y\| \geq \frac{1}{4\gamma} \text{diam } S\}.$$

Let $R := \frac{1}{4\gamma} \text{diam } S$. Then, by (3.27) and (3.28),

$$\sum \{(\text{diam } Q)^{2-p} : Q \in G_2(S)\} \leq C(\gamma, p) \int_{\|x_0 - y\| \geq R} \|x_0 - y\|^{-p} dy \leq C(\gamma, p) R^{2-p}.$$

We obtain

$$\sum \{(\text{diam } Q)^{2-p} : Q \in G_2(S)\} \leq C(\gamma, p)(\text{diam } S)^{2-p}.$$

This estimate and inequality (3.26) imply the required inequality (3.24).

In turn (3.23) and (3.24) imply the following inequality:

$$I_1 \leq C(\gamma, p) \sum_{S \in \tilde{\mathcal{S}}} \int_{V\{S\}} H(z) dz.$$

Recall that

$$V\{S\} := \bigcup \{K \in \mathcal{K} : c_K \in S\}.$$

Since the families \mathcal{S} and \mathcal{K} consist of pairwise disjoint sets, the sets of the family $\{V\{S\} : S \in \mathcal{S}\}$ are pairwise disjoint as well. Hence

$$I_1 \leq C(\gamma, p) \int_{U\{\mathcal{K}\}} H(z) dz.$$

In the same way we prove that

$$I_2 \leq C(\gamma, p) \int_{U\{\mathcal{K}\}} H(z) dz.$$

Combining these inequalities with (3.22) and (3.21) we obtain the statement of the proposition. \blacksquare

Let us finish the proof of the necessity of part (ii) of Theorem 1.2.

Let $F \in L_p^2(\mathbf{R}^2)$ and let $F|_E = f$. Let us estimate the quantity

$$A(f) = \sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{2-p} S_p(f : S'_Q, S''_Q; \mathcal{K})}{\{(\text{diam } S'_Q)^{2-p} + \sigma_p(S'_Q; \mathcal{K})\} \{(\text{diam } S''_Q)^{2-p} + \sigma_p(S''_Q; \mathcal{K})\}}.$$

Since $F|_E = f$, we have $S_p(f : Q', Q''; \mathcal{K}) = S_p(F : Q', Q''; \mathcal{K})$ so that $A(f) = A(F)$. Let $q := (p+2)/2$. Then, by Proposition 3.9,

$$A(f) \leq C \left\{ \int_{U\{\mathcal{Q}\}} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz + \int_{U\{\mathcal{Q}\}} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz \right\} \leq C \int_{\mathbf{R}^2} \mathcal{M}[(\nabla^2 F)^q]^{\frac{p}{q}} dz$$

where C is a constant depending only on p and γ .

Since $p/q > 1$, by the Hardy-Littlewood maximal theorem,

$$A(f) \leq C \int_{\mathbf{R}^2} [(\nabla^2 F)^q]^{\frac{p}{q}} dz = C \int_{\mathbf{R}^2} (\nabla^2 F)^p dz = C \|F\|_{L_p^2(\mathbf{R}^2)}^p$$

proving again that for every $F \in L_p^2(\mathbf{R}^2) \cap C^1(\mathbf{R}^2)$ such that $F|_E = f$ the condition (ii) of Theorem 1.2 holds with $\lambda = C(p, \gamma) \|F\|_{L_p^2(\mathbf{R}^2)}^p$.

The necessity part of Theorem 1.2 is completely proved. \blacksquare

4. Lacunae of Whitney cubes.

We turn to the proof of the sufficiency part of Main Theorem 1.2.

We prove the sufficiency in several steps. At the first step we present a modification of the Whitney extension method which is based on the notion of a “*lacuna of Whitney squares*”. We have briefly described this object in Section 2. This notion enables us to identify and characterize all possible “holes” in the set E . In this section we give main definitions and describe several main properties of lacunae.

As we have mentioned in Section 2, *in this and the next two sections E is an arbitrary closed subset of \mathbf{R}^n* . We equip \mathbf{R}^n with the uniform norm $\|x\| := \max\{|x_i| : i = 1, \dots, n\}$.

As usual the word “cube” will mean a closed cube in \mathbf{R}^n whose sides are parallel to the coordinate axes. We will use the same notation for cubes and distances between sets as in two dimensional case.

4.1. Whitney cubes and lacunae of Whitney cubes. First let us recall the notion of a Whitney cube. Since E is a closed set, the set $\mathbf{R}^n \setminus E$ is open so that it admits a Whitney decomposition W_E into a family of non-overlapping cubes. In the next theorem we present the main properties of this decomposition, see, e.g. [43] or [23].

Theorem 4.1 *$W_E = \{Q_k\}$ is a countable family of non-overlapping cubes such that*

- (i). $\mathbf{R}^n \setminus E = \cup\{Q : Q \in W_E\}$;
- (ii). *For every cube $Q \in W_E$ we have*

$$(4.1) \quad \text{diam } Q \leq \text{dist}(Q, E) \leq 4 \text{diam } Q.$$

We are also needed certain additional properties of Whitney cubes which we present in the next lemma. These properties easily follow from constructions of Whitney decomposition presented in [43] and [23].

Given a cube $Q \subset \mathbf{R}^n$ let $Q^* := \frac{9}{8}Q$.

Lemma 4.2 (1). *If $Q, K \in W_E$ and $Q^* \cap K^* \neq \emptyset$, then*

$$\frac{1}{4} \text{diam } Q \leq \text{diam } K \leq 4 \text{diam } Q.$$

(2). *For every cube $K \in W_E$ there are at most $N = N(n)$ cubes from the family*

$$W_E^* := \{Q^* : Q \in W_E\}$$

which intersect K^ .*

(3). *If $Q, K \in W_E$, then $Q^* \cap K^* \neq \emptyset$ if and only if $Q \cap K \neq \emptyset$.*

Note that inequality (4.1) implies the following property of Whitney cubes:

$$(4.2) \quad (9Q) \cap E \neq \emptyset \quad \text{for every } Q \in W_E.$$

By LW_E we denote a subfamily of Whitney cubes satisfying the following condition:

$$(4.3) \quad (10Q) \cap E = (90Q) \cap E.$$

Then we introduce a binary relation \sim on LW_E : for every $Q_1, Q_2 \in LW_E$

$$Q_1 \sim Q_2 \iff (10Q_1) \cap E = (10Q_2) \cap E.$$

It can be easily seen that \sim satisfies the axioms of equivalence relations, i.e., it is reflexive, symmetric and transitive. Given a cube $Q \in LW_E$ by

$$[Q] := \{K \in LW_E : K \sim Q\}$$

we denote the equivalence class of Q . We refer to this equivalence class as *a true lacuna* with respect to the set E .

Let

$$\tilde{\mathcal{L}}_E = LW_E / \sim = \{[Q] : Q \in LW_E\}$$

be the corresponding quotient set of LW_E by \sim , i.e., the set of all possible equivalence classes (lacunae) of LW_E by \sim .

Thus for every pair of Whitney cubes $Q_1, Q_2 \in W_E$ which belong to a true lacuna $L \in \tilde{\mathcal{L}}_E$ we have

$$(4.4) \quad (10Q_1) \cap E = (90Q_1) \cap E = (10Q_2) \cap E = (90Q_2) \cap E.$$

By V_L we denote the associated set of the lacuna L

$$(4.5) \quad V_L := (90Q) \cap E.$$

Here Q is an arbitrary cube from L . By (4.4), any choice of a cube $Q \in L$ provides the same set V_L so that V_L is well-defined. Also note that for each cube Q which belong to a true lacuna L we have $V_L = (10Q) \cap E$.

We extend the family $\tilde{\mathcal{L}}_E$ of true lacunae to a family \mathcal{L}_E of *all lacunae* in the following way. Suppose that $Q \in W_E \setminus LW_E$, see (4.3), i.e.,

$$(4.6) \quad (10Q) \cap E \neq (90Q) \cap E.$$

In this case to the cube Q we assign a lacuna $L := \{Q\}$ consisting of a unique cube - the cube Q . We also put $V_L := (90Q) \cap E$ as in (4.5).

We refer to such a lacuna $L := \{Q\}$ as an *elementary lacuna* with respect to the set E . By $\hat{\mathcal{L}}_E$ we denote the family of all elementary lacunae with respect to E :

$$\hat{\mathcal{L}}_E := \{L = \{Q\} : Q \in W_E \setminus LW_E\}$$

We note that property (4.6) implies the existence of a point

$$a \in (E \setminus (10Q)) \cap (90Q).$$

On the other hand, by (4.2), there exists a point

$$b \in (9Q) \cap E.$$

Hence

$$\|a - b\| \geq r_Q = (1/2) \text{diam } Q$$

so that

$$(4.7) \quad \text{diam } V_L = \text{diam}((90Q) \cap E) \geq \frac{1}{2} \text{diam } Q$$

provided

$$L = \{Q\} \in \hat{\mathcal{L}}_E$$

is an elementary lacuna.

Finally, by \mathcal{L}_E we denote the family of all lacunae with respect to E :

$$\mathcal{L}_E = \tilde{\mathcal{L}}_E \cup \hat{\mathcal{L}}_E.$$

4.2. Main properties of the lacunae.

Lemma 4.3 *Let $L \in \mathcal{L}_E$ be a lacuna and let $Q \in L$. Then*

$$\text{dist}(Q, V_L) = \text{dist}(Q, E).$$

Proof. Recall that for each Whitney cube $Q \in W_E$ we have $(9Q) \cap E \neq \emptyset$. Hence

$$\text{dist}(Q, E) = \text{dist}(Q, (9Q) \cap E) = \text{dist}(Q, (90Q) \cap E).$$

But $V_L = (90Q) \cap E$ for each $Q \in L$, see (4.5), and the proof is finished. ■

Proposition 4.4 *Let $L \in \mathcal{L}_E$ be a lacuna. Then*

$$(4.8) \quad \frac{1}{90} \text{diam } V_L \leq \inf_{Q \in L} \text{diam } Q \leq \gamma \text{diam } V_L$$

where γ is an absolute constant.

Proof. Suppose that $\text{diam } V_L = 0$ so that $V_L = \{A_L\}$ where A_L is a point in E . Then for each $\varepsilon > 0$ the ε -neighborhood of A_L contains a cube from L . This proves (4.8) in the case under consideration.

Let us assume that $\text{diam } V_L > 0$. If $L \in \hat{\mathcal{L}}$ is an elementary lacuna, then $L = \{Q\}$ where $Q \in W_E \setminus LW_E$. In this case the statement of the proposition follows from (4.5) and inequality (4.7) with $\gamma = 2$.

Let $L \in \tilde{\mathcal{L}}_E$ be a true lacuna. We recall that in this case every cube $Q \in L$ satisfies the condition

$$(4.9) \quad V_L = (10Q) \cap E = (90Q) \cap E.$$

Hence $\text{diam } V_L \leq 10 \text{diam } Q$ for every $Q \in L$ proving the first inequality in (4.8).

Prove the second inequality in (4.8) with $\gamma := 600$. Suppose that it is not true, i.e.,

$$(4.10) \quad \text{diam } Q > \gamma \text{diam } V_L \quad \text{for every } Q \in L.$$

Let $m = 30$. Fix a cube $Q \in L$ and prove that there exists a cube $K \in L$ such that $\text{diam } K \leq \frac{1}{m} \text{diam } Q$.

Note that, by Lemma 4.3, $\text{dist}(Q, V_L) = \text{dist}(Q, E)$ and, by Theorem 4.1, $\text{diam } Q \leq \text{dist}(Q, E)$. Hence

$$\text{diam } Q \leq \text{dist}(Q, V_L) \leq \text{dist}(x, V_L) \quad \text{for every } x \in Q.$$

Thus $\text{dist}(\cdot, V_L)$ is a continuous function on \mathbf{R}^n which takes the value 0 on V_L . Since the values of this function on Q are at least $\text{diam } Q$, there exists a point $x_0 \in \mathbf{R}^n \setminus E$ such that

$$(4.11) \quad \text{dist}(x_0, V_L) = \frac{1}{m} \text{diam } Q.$$

Prove that

$$(4.12) \quad \text{dist}(x_0, V_L) = \text{dist}(x_0, E).$$

In fact, by (4.9) and (4.11),

$$x_0 \in (10 + 2/m)Q.$$

This and (4.9) imply the following inequality

$$\text{dist}(x_0, E \setminus V_L) = \text{dist}(x_0, E \setminus (90Q)) \geq (90 - (10 + 2/m))r_Q = (80 - 2/m)r_Q.$$

Hence

$$(4.13) \quad \text{dist}(x_0, E \setminus V_L) \geq (40 - 1/m) \text{diam } Q.$$

Since $1/m < 40 - 1/m$, this inequality and equality (4.11) imply that

$$\text{dist}(x_0, V_L) < \text{dist}(x_0, E \setminus V_L)$$

proving (4.12).

Let $K \in W_E$ be a Whitney cube containing x_0 . Then, by Theorem 4.1,

$$(4.14) \quad \text{diam } K \leq \text{dist}(K, E) \leq 4 \text{diam } K.$$

Prove that

$$(4.15) \quad \frac{1}{5m} \text{diam } Q \leq \text{diam } K \leq \frac{1}{m} \text{diam } Q.$$

Since $x_0 \in K$, we have

$$\text{dist}(x_0, V_L) = \text{dist}(x_0, E) \leq \text{dist}(K, E) + \text{diam } K \leq 5 \text{diam } K$$

so that, by (4.11), $(1/5m) \text{diam } Q \leq \text{diam } K$. Also,

$$\text{diam } K \leq \text{dist}(K, E) \leq \text{dist}(x_0, E) = \text{dist}(x_0, V_L)$$

so that, by (4.11), $\text{diam } K \leq (1/m) \text{diam } Q$, and (4.15) is proved.

Prove that $K \in L$, i.e.,

$$(4.16) \quad V_L = (10K) \cap E = (90K) \cap E.$$

Since $\text{dist}(\cdot, V_L)$ is a Lipschitz function, by (4.11), for every $y \in K$ we have

$$\text{dist}(y, V_L) \leq \|x_0 - y\| + \text{dist}(x_0, V_L) \leq \text{diam } K + \frac{1}{m} \text{diam } Q$$

so that, by (4.15),

$$\text{dist}(y, V_L) \leq \frac{2}{m} \text{diam } Q.$$

On the other hand, by (4.13) and (4.15),

$$\begin{aligned} \text{dist}(y, E \setminus V_L) &\geq \text{dist}(x_0, E \setminus V_L) - \text{diam } K \\ &\geq (40 - \frac{1}{m}) \text{diam } Q - \frac{1}{m} \text{diam } Q = (40 - \frac{2}{m}) \text{diam } Q. \end{aligned}$$

Since $\frac{2}{m} \leq 40 - \frac{2}{m}$ for $m \geq 1$, we obtain

$$\text{dist}(y, V_L) < \text{dist}(y, E \setminus V_L)$$

proving that

$$\text{dist}(y, V_L) = \text{dist}(y, E) \quad \text{for every } y \in K.$$

Hence

$$(4.17) \quad \text{dist}(K, E) = \text{dist}(K, V_L).$$

By (4.17), there exists a point $b \in V_L$ which is a point nearest to K on E . Then, by (4.14),

$$\text{dist}(b, K) = \text{dist}(K, E) \leq 4 \text{diam } K.$$

Let $K = Q(c_K, r_K)$, i.e., c_K is the center of K and $r_K = \frac{1}{2} \text{diam } K$. Then for each $z \in V_L$

$$\|z - c_K\| \leq \|z - b\| + \text{dist}(b, K) + r_K \leq \text{diam } V_L + 4 \text{diam } K + r_K$$

so that

$$(4.18) \quad \|z - c_K\| \leq \text{diam } V_L + 9r_K.$$

But, by assumption (4.10), $\text{diam } V_L < (1/\gamma) \text{diam } Q$, so that, by (4.15),

$$\|z - c_K\| \leq (1/\gamma) \text{diam } Q + 9r_K \leq (5m/\gamma) \text{diam } K + 9r_K \leq (9 + (10m/\gamma))r_K$$

proving that

$$V_L \subset (9 + (10m/\gamma))K.$$

Hence

$$(4.19) \quad V_L \subset 10K$$

provided $10m < \gamma$. Since $V_L \subset E$, this shows that $V_L \subset (10K) \cap E$.

Prove that

$$(4.20) \quad 10K \subset 10Q.$$

In fact, since $Q \in W_E$ we have $(9Q) \cap E \neq \emptyset$. Combining this with (4.9) we obtain

$$(9Q) \cap E = (9Q) \cap V_L \neq \emptyset$$

so that, by (4.19),

$$(10K) \cap (9Q) \neq \emptyset.$$

But, by (4.15), $\text{diam } K \leq \frac{1}{m} \text{diam } Q$, so that

$$10K \subset (9 + 20/m)Q.$$

Since $m \geq 20$, we obtain the required imbedding (4.20).

By dilation, from (4.20) we have $90K \subset 90Q$.

Finally, by (4.9),

$$V_L \subset (10K) \cap E \subset (10Q) \cap E = V_L$$

and

$$V_L \subset (90K) \cap E \subset (90Q) \cap E = V_L$$

proving (4.16). Hence $K \in L$.

Thus we have proved that for every $Q \in L$ there exists a cube $K \in L$ such that $\text{diam } K \leq \frac{1}{m} \text{diam } Q$. Hence

$$\inf_{Q \in L} \text{diam } Q = 0.$$

But, by the assumption (4.10), $\text{diam } V_L < (1/\gamma) \text{diam } Q$ so that

$$\inf_{Q \in L} \text{diam } Q \geq \gamma \text{diam } V_L$$

which implies that $\text{diam } V_L = 0$. But $\text{diam } V_L > 0$, a contradiction. Thus the second inequality in (4.8) holds.

The proposition is completely proved. ■

Given a lacuna $L \in \mathcal{L}_E$ we introduce a pair of two cubes characterizing the “size” of this lacuna - a cube Q_L of the minimal diameter, and the cube $Q^{(L)}$ of the maximal diameter. These cubes not always exist. Let us describe conditions for their existence and present main properties of these cubes.

Proposition 4.5 *Let $L \in \mathcal{L}_E$ be a lacuna. If $\text{diam } V_L > 0$, then there exists a cube $Q_L \in L$ such that*

$$\text{diam } Q_L = \min\{\text{diam } Q : Q \in L\}.$$

Furthermore,

$$(4.21) \quad \frac{1}{90} \text{diam } V_L \leq \text{diam } Q_L \leq \gamma \text{diam } V_L$$

where γ is an absolute constant.

Proof. Inequality (4.21) follows from Proposition 4.4 so we turn to the proof of the existence of the cube Q_L .

Of course, its trivial for an elementary lacuna so we may assume that $L \in \tilde{\mathcal{L}}$ is a true lacuna. Fix a cube $\tilde{Q} \in L$ and consider a family of cubes

$$J_{\tilde{Q}} = \{K \in L : \text{diam } K \leq \text{diam } \tilde{Q}\}.$$

Prove that $J_{\tilde{Q}}$ is a *finite* family of cubes. In fact, since L is a true lacuna, $V_L = (10\tilde{Q}) \cap E$ and

$$V_L = (10K) \cap E \quad \text{for every } K \in J_{\tilde{Q}}$$

so that $(10K) \cap (10\tilde{Q}) \neq \emptyset$. But $\text{diam } K \leq \text{diam } \tilde{Q}$ which implies that $K \subset 10K \subset 30\tilde{Q}$.

Thus $\cup\{K : K \in J_{\tilde{Q}}\} \subset 30\tilde{Q}$. By Proposition 4.4, $\text{diam } K \geq \text{diam } V_L/90$ for every $K \in J_{\tilde{Q}}$. Since $J_{\tilde{Q}} \subset W_E$, the cubes of this family are non-overlapping. These properties of $J_{\tilde{Q}}$ immediately implies that $\#J_{\tilde{Q}} < \infty$.

Clearly,

$$\inf_{Q \in L} \text{diam } Q = \inf_{K \in J_{\tilde{Q}}} \text{diam } K = \min_{K \in J_{\tilde{Q}}} \text{diam } K$$

so that the minimum of $\text{diam } Q$ is attained on L proving the proposition. ■

Summarizing inequality (4.21) and inequality (4.7), we obtaining the following

Corollary 4.6 *For every $L \in \mathcal{L}_E$ such that $\text{diam } V_L > 0$ we have*

$$\text{diam } Q_L \leq \eta \text{diam}((\theta Q_L) \cap E)$$

where $\theta = 90$ and η is an absolute constant.

Proposition 4.7 *For every lacuna $L \in \mathcal{L}_E$ we have*

$$(4.22) \quad \frac{1}{\gamma} \text{dist}(V_L, E \setminus V_L) \leq \sup_{Q \in L} \text{diam } Q \leq 40 \text{dist}(V_L, E \setminus V_L)$$

where γ is an absolute constant.

Proof. For an elementary lacuna L the statement of the proposition follows from inequality (4.7).

Let $L \in \tilde{\mathcal{L}}$ be a true lacuna so that for every cube $Q \in L$ we have

$$(4.23) \quad V_L = (10Q) \cap E = (90Q) \cap E.$$

Hence

$$\text{dist}(V_L, E \setminus V_L) \geq \text{dist}(10Q, \mathbf{R}^n \setminus 90Q) = 80r_Q = 40 \text{diam } Q$$

proving the second inequality in (4.22).

Prove the first inequality. This inequality is trivial whenever $\sup_{Q \in L} \text{diam } Q = \infty$. Suppose that

$$(4.24) \quad \sup_{Q \in L} \text{diam } Q < \infty.$$

Let us assume that

$$(4.25) \quad \text{dist}(V_L, E \setminus V_L) > \gamma \sup_{Q \in L} \text{diam } Q$$

with

$$\gamma := 10^4.$$

Prove that this assumption contradicts (4.24). Let

$$m := 100.$$

Fix a cube $Q \in L$ and construct a cube $K \in L$ such that

$$\text{diam } K \geq \frac{m}{5} \text{ diam } Q.$$

By the assumption (4.25),

$$(4.26) \quad \text{dist}(V_L, E \setminus V_L) > \gamma \text{ diam } Q.$$

Note that the function $\text{dist}(\cdot, V_L)$ is continuous and equals 0 on V_L . Besides it is strictly greater than $\gamma \text{ diam } Q$ on $E \setminus V_L$. Since $0 < m < \gamma$, there exists a point $x_0 \in \mathbf{R}^n \setminus E$ such that

$$(4.27) \quad \text{dist}(x_0, V_L) = m \text{ diam } Q.$$

Let $K \in W_E$ be a Whitney cube which contains x_0 . Prove that

$$\text{diam } K \geq \frac{m}{5} \text{ diam } Q \quad \text{and} \quad K \in L.$$

First let us show that

$$(4.28) \quad \text{dist}(x_0, V_L) = \text{dist}(x_0, E).$$

Note that, by (4.23), $V_L \subset 10Q$ so that $\text{diam } V_L \leq 10 \text{ diam } Q$.

By (4.26) and (4.27),

$$\begin{aligned} \text{dist}(x_0, E \setminus V_L) &\geq \text{dist}(V_L, E \setminus V_L) - \text{diam } V_L - \text{dist}(x_0, V_L) \\ &> \gamma \text{ diam } Q - 10 \text{ diam } Q - m \text{ diam } Q. \end{aligned}$$

Hence,

$$(4.29) \quad \text{dist}(x_0, E \setminus V_L) > (\gamma - m - 10) \text{ diam } Q > m \text{ diam } Q.$$

Combining this inequality with (4.27) we conclude that $\text{dist}(x_0, V_L) < \text{dist}(x_0, E \setminus V_L)$ which proves (4.28).

Since $K \in W_E$, by Theorem 4.1,

$$\text{diam } K \leq \text{dist}(K, E) \leq 4 \text{ diam } K.$$

Prove that

$$(4.30) \quad \frac{m}{5} \text{ diam } Q \leq \text{diam } K \leq m \text{ diam } Q.$$

Since $x_0 \in K$, we have

$$\text{dist}(x_0, V_L) = \text{dist}(x_0, E) \leq \text{dist}(K, E) + \text{diam } K \leq 5 \text{ diam } K$$

so that, by (4.27), $m \operatorname{diam} Q \leq 5 \operatorname{diam} K$. Conversely,

$$\operatorname{diam} K \leq \operatorname{dist}(K, E) \leq \operatorname{dist}(x_0, E) = \operatorname{dist}(x_0, V_L)$$

so that, by (4.27), $\operatorname{diam} K \leq m \operatorname{diam} Q$, and (4.30) is proved.

Prove that $K \in L$, i.e.,

$$(4.31) \quad V_L = (10K) \cap E = (90K) \cap E.$$

First let us show that

$$(4.32) \quad \operatorname{dist}(K, E) = \operatorname{dist}(K, V_L).$$

Let $y \in K$. Then, by (4.30),

$$\operatorname{dist}(y, V_L) \leq \|x_0 - y\| + \operatorname{dist}(x_0, V_L) \leq \operatorname{diam} K + m \operatorname{diam} Q$$

so that, by (4.30),

$$\operatorname{dist}(y, V_L) \leq 2m \operatorname{diam} Q.$$

On the other hand, by (4.29) and (4.30),

$$\begin{aligned} \operatorname{dist}(y, E \setminus V_L) &\geq \operatorname{dist}(x_0, E \setminus V_L) - \operatorname{diam} K \\ &\geq (\gamma - m - 20) \operatorname{diam} Q - m \operatorname{diam} Q \\ &= (\gamma - 2m - 20) \operatorname{diam} Q > 2m \operatorname{diam} Q. \end{aligned}$$

Hence

$$\operatorname{dist}(y, V_L) < \operatorname{dist}(y, E \setminus V_L) \quad \text{for every } y \in K$$

proving (4.32).

Repeating the proof of inequality (4.18) from Proposition 4.4 we conclude that for every $z \in V_L$

$$\|z - c_K\| \leq \operatorname{diam} V_L + 9r_K.$$

(Recall that $K = Q(c_K, r_K)$.)

Since $V_L \subset 10Q$, we have $\operatorname{diam} V_L \leq 10 \operatorname{diam} Q$ so that

$$\|z - c_K\| \leq 10 \operatorname{diam} Q + 9r_K.$$

Combining this inequality with (4.30) we obtain

$$\|z - c_K\| \leq 10(5 \operatorname{diam} K/m) + 9r_K = (9 + 50/m)r_K < 10r_K.$$

Hence $V_L \subset 10Q$.

Prove that

$$(4.33) \quad (90K) \cap (E \setminus V_L) = \emptyset.$$

In fact, since $90K \supset V_L$, we have

$$\operatorname{dist}(90K, E \setminus V_L) \geq \operatorname{dist}(V_L, E \setminus V_L) - \operatorname{diam}(90K)$$

so that, by the assumption (4.26),

$$\text{dist}(90K, E \setminus V_L) \geq \gamma \text{diam } Q - 90 \text{diam } K.$$

Since $\text{diam } Q \geq \text{diam } K/m$, see (4.30), we obtain

$$\text{dist}(90K, E \setminus V_L) \geq (\gamma/m - 90) \text{diam } K.$$

Since $\gamma > 90m$, we have $\text{dist}(90K, E \setminus V_L) > 0$ proving (4.33).

Hence $(90K) \cap E = (90K) \cap V_L = V_L$. Finally,

$$V_L \subset (10K) \cap E \subset (90K) \cap E = V_L$$

proving (4.31). Thus $K \in L$.

We have proved that for every cube $Q \in L$ there exists a cube $K \in L$ such that

$$\text{diam } K \geq \frac{m}{5} \text{diam } Q.$$

Hence $\sup\{\text{diam } Q : Q \in L\} = \infty$ which contradicts the condition (4.24).

Thus inequality (4.25) is not true which proves the first inequality in (4.22) and the proposition. \blacksquare

Let $L \in \mathcal{L}_E$ be a lacuna. By U_L we denote the union of all cubes which belong to the lacuna:

$$U_L := \cup\{Q : Q \in L\}.$$

By $\text{diam } L$ we denote the diameter of the set U_L :

$$\text{diam } L := \text{diam } U_L = \sup\{\|a - b\| : a, b \in U_L\}.$$

We say that L is bounded if $\text{diam } L < \infty$. If $\text{diam } L = \infty$ we say that L is an unbounded lacuna.

Proposition 4.8 (i). For every lacuna $L \in \mathcal{L}_E$

$$(4.34) \quad \text{diam } L \sim \sup\{\text{diam } Q : Q \in L\} \sim \text{dist}(V_L, E \setminus V_L)$$

with absolute constants in the equivalences;

(ii). If E is an unbounded set then every lacuna $L \in \mathcal{L}_E$ is bounded;

(iii). If E is bounded, there exists the unique unbounded lacuna $L^{\max} \in \mathcal{L}_E$. The lacuna L^{\max} is a true lacuna for which $V_{L^{\max}} = E$.

Proof. (i). The second equivalence in (4.34) follows from Proposition 4.7.

Clearly $\sup\{\text{diam } Q : Q \in L\} \leq \text{diam } L$. Prove that

$$(4.35) \quad \text{diam } L \leq 30 \sup\{\text{diam } Q : Q \in L\}.$$

Of course this inequality is trivial for elementary lacunae. Consider a true lacuna $L \in \tilde{\mathcal{L}}$. Let cubes $K, K' \in L$ and let $a \in K$ and $a' \in K'$. Suppose that $\text{diam } K' \leq \text{diam } K$. Since L is a true lacuna,

$$V_L = (10K) \cap E = (10K') \cap E$$

so that $(10K) \cap (10K') \neq \emptyset$. Since $\text{diam } K' \leq \text{diam } K$, we have

$$K' \subset 10K' \subset 30K.$$

Hence $a, a' \in 30K$ so that

$$\|a - a'\| \leq \text{diam}(30K) = 30 \text{diam } K$$

proving (4.35) and part (i) of the proposition.

(ii). Let E be an unbounded set. Of course, in this case every elementary lacuna of E is bounded. If L is a true lacuna, by part (i) of the proposition,

$$(4.36) \quad \text{diam } L \sim \text{dist}(V_L, E \setminus V_L).$$

Since V_L is a bounded set and E is an unbounded set, $\text{dist}(V_L, E \setminus V_L) < \infty$ proving that $\text{diam } L < \infty$.

(iii). Let E be a bounded set and let $L \in \tilde{\mathcal{L}}$ be a true lacuna. If $V_L \neq E$, then

$$\text{dist}(V_L, E \setminus V_L) < \infty$$

so that, by (4.36), $\text{diam } L < \infty$.

On the other hand, the unique true lacuna L^{\max} such that $V_{L^{\max}} = E$ contains cubes of arbitrarily big sizes so that $\text{diam } L^{\max} = \infty$. ■

Proposition 4.9 *Let $L \in \mathcal{L}_E$ be a bounded lacuna. Then there exists a cube $Q^{(L)} \in L$ such that*

$$\text{diam } Q^{(L)} = \max\{\text{diam } K : K \in L\}.$$

Furthermore,

$$(4.37) \quad \text{diam } Q^{(L)} \sim \text{diam } L \sim \text{dist}(V_L, E \setminus V_L),$$

and

$$(4.38) \quad V_L \cup U_L \subset \gamma Q^{(L)}.$$

Here the constant γ and constants in the equivalences of (4.37) are absolute .

Proof. Clearly equivalences (4.37) follow from (4.34), so we turn to the proof of the existence of the cube $Q^{(L)}$. Fix a cube $\tilde{Q} \in L$ and introduce a family of cubes

$$I_{\tilde{Q}} = \{K \in L : \text{diam } K \geq \text{diam } \tilde{Q}\}.$$

Note that the diameters of the cubes from $I_{\tilde{Q}}$ are at least $\text{diam } \tilde{Q}$, these cubes are non-overlapping and lie in a bounded set so that the family $I_{\tilde{Q}}$ is finite. Therefore there exists a cube $Q^{(L)} \in I_{\tilde{Q}}$ such that $\text{diam } Q^{(L)} = \max\{\text{diam } K : K \in I_{\tilde{Q}}\}$. But

$$\max\{\text{diam } K : K \in I_{\tilde{Q}}\} = \max\{\text{diam } K : K \in L\}$$

proving that

$$\text{diam } Q^{(L)} = \max\{\text{diam } K : K \in L\}.$$

It remains to prove inclusion (4.38).

Let $\gamma = 270$. We know that $V_L = (90Q) \cap E$ for every cube $Q \in L$. In particular, $V_L \subset 90Q^{(L)} \subset \gamma Q^{(L)}$.

Since $V_L \subset (90Q) \cap (90Q^{(L)})$, we conclude that $(90Q) \cap (90Q^{(L)}) \neq \emptyset$. But $\text{diam } Q \leq \text{diam } Q^{(L)}$ so that $Q \subset 270Q^{(L)}$ proving that

$$U_L = \cup\{Q : Q \in L\} \subset \gamma Q^{(L)}. \quad \blacksquare$$

Proposition 4.10 *Let $L \in \mathcal{L}_E$ be a lacuna and let $Q \in L$. Suppose that there exist a lacuna $L' \in \mathcal{L}_E$, $L \neq L'$, and a cube $Q' \in L'$ such that $Q \cap Q' \neq \emptyset$. Then:*

- (i). *If L is a true lacuna, then L' is an elementary lacuna, i.e., $L' \in \hat{\mathcal{L}}_E = \mathcal{L}_E \setminus \tilde{\mathcal{L}}_E$;*
- (ii). *Either*

$$(4.39) \quad \text{diam } Q \sim \text{diam } V_L \sim \text{diam } Q_L$$

or

$$(4.40) \quad \text{diam } Q \sim \text{dist}(V_L, E \setminus V_L) \sim \text{diam } Q^{(L)}$$

with absolute constants in the equivalences.

Proof. (i). Since $L \in \tilde{\mathcal{L}}_E$ is a true lacuna, for every $Q \in L$ we have

$$V_L = (10Q) \cap E = (90Q) \cap E.$$

Prove that $L' \in \hat{\mathcal{L}} = \mathcal{L} \setminus \tilde{\mathcal{L}}_E$. In fact, if $L' \in \tilde{\mathcal{L}}_E$, then

$$V_{L'} = (10Q') \cap E = (90Q') \cap E.$$

Since $L \neq L'$, we have $V_L \neq V_{L'}$. We know that $Q, Q' \in W_E$ and $Q \cap Q' \neq \emptyset$ so that, by Lemma 4.2,

$$(4.41) \quad 1/4 \text{diam } Q \leq \text{diam } Q' \leq 4 \text{diam } Q.$$

Hence $Q' \subset 9Q$ so that $10Q' \subset 90Q$ proving that

$$V_{L'} = (10Q') \cap E \subset (90Q) \cap E = V_L.$$

In the same way we prove that $V_L \subset V_{L'}$. Hence $V_L = V_{L'}$, a contradiction.

(ii). Note that the second equivalence in (4.39) follows from (4.21). In turn the second equivalence in (4.40) follows from (4.37). Thus we have to prove that either the first equivalence in (4.39) or the first equivalence in (4.40) holds.

Note that if $L \in \hat{\mathcal{L}}$, i.e., L is an elementary lacuna, then, by (4.7), equivalence (4.39) holds. Thus we may assume that L is a true lacuna.

In part (i) we have proved that in this case $L' \in \hat{\mathcal{L}} = \mathcal{L}_E \setminus \tilde{\mathcal{L}}_E$ so that the cube Q' is the unique cube which belongs to the lacuna L' . Note that, by (4.7),

$$\text{diam } Q' \leq 2 \text{diam}((90Q') \cap E).$$

Also recall that $Q' \subset 9Q$, so that $10Q' \subset 90Q$. Combining this with (4.41), we obtain

$$(4.42) \quad \text{diam } Q \leq 4 \text{diam } Q' \leq 8 \text{diam}((90Q') \cap E) \leq 8 \text{diam}((\gamma_1 Q) \cap E)$$

with $\gamma_1 := 810$. Now consider two cases.

The first case:

$$\text{diam } Q \leq 8 \text{diam } V_L.$$

In this case the first equivalence in (4.39) holds. In fact, since $V_L = (10Q) \cap E$, we have $\text{diam } V_L \leq 10 \text{diam } Q$ so that

$$\frac{1}{8} \text{diam } Q \leq \text{diam } V_L \leq 10 \text{diam } Q.$$

The second case:

$$(4.43) \quad \text{diam } V_L < \frac{1}{8} \text{diam } Q.$$

Prove that in this case equivalence (4.40) holds. In fact, by (4.42), there exist points $a, b \in (\gamma_1 Q) \cap E$ such that

$$\frac{1}{8} \text{diam } Q \leq \|a - b\|$$

so that, by (4.43), either a or b does not belong to V_L . Assume that $a \notin V_L$. Then

$$\text{dist}(V_L, E \setminus V_L) \leq \text{dist}(a, V_L).$$

Since $V_L \subset 10Q \subset \gamma_1 Q$ and $a \in \gamma_1 Q$, we conclude that

$$\text{dist}(a, V_L) \leq \text{diam}(\gamma_1 Q) = \gamma_1 \text{diam } Q$$

proving that

$$(4.44) \quad \text{dist}(V_L, E \setminus V_L) \leq \gamma_1 \text{diam } Q.$$

On the other hand, by (4.34),

$$\text{diam } Q \leq \sup_{K \in L} \text{diam } K \leq \gamma_2 \text{dist}(V_L, E \setminus V_L)$$

with an absolute constant γ_2 . This inequality and inequality (4.44) prove equivalence (4.40) and finish the proof of the proposition. \blacksquare

Proposition 4.11 *Let $L \in \mathcal{L}_E$ be a lacuna and let*

$$\mathcal{I}_L := \{K \in W_E \setminus L : \exists Q \in L \text{ such that } K \cap Q \neq \emptyset\}.$$

Then $\#\mathcal{I}_L \leq \gamma(n)$.

Proof. Let $K \in \mathcal{I}_L$, i.e., K belongs to a lacuna L' , $L' \neq L$, and $K \cap Q_K \neq \emptyset$ for some cube $Q_K \in L$. Then, by Proposition 4.10, either $\text{diam } Q_K \sim \text{diam } V_L$ or

$$\text{diam } Q_K \sim \text{dist}(V_L, E \setminus V_L).$$

Since

$$(4.45) \quad 1/4 \text{diam } Q_K \leq \text{diam } K \leq 4 \text{diam } Q_K,$$

see Lemma 4.2, we conclude that either

$$(4.46) \quad \text{diam } K \sim \text{diam } V_L$$

or

$$(4.47) \quad \text{diam } K \sim \text{dist}(V_L, E \setminus V_L)$$

(with absolute constants in the equivalences).

Let us denote by $\mathcal{I}_L^{(1)}$ a subfamily of \mathcal{I}_L consisting of those cubes K which satisfy (4.46). By $\mathcal{I}_L^{(2)}$ we denote those cubes $K \in \mathcal{I}_L$ which satisfy inequality (4.47). Then $\mathcal{I}_L = \mathcal{I}_L^{(1)} \cup \mathcal{I}_L^{(2)}$ so that

$$\#\mathcal{I}_L \leq \#\mathcal{I}_L^{(1)} + \#\mathcal{I}_L^{(2)}.$$

Prove that $\#\mathcal{I}_L^{(1)} \leq \gamma(n)$. To this end we put $R_1 := \text{diam } V_L$. Then

$$(4.48) \quad \text{diam } K \sim R_1 \text{ for all } K \in \mathcal{I}_L^{(1)}.$$

Since $Q_K \in L$,

$$V_L = (10Q_K) \cap E = (90Q_K) \cap E.$$

Let us fix a point $x_0 \in V_L$. Then for every $y \in K$ we have

$$\|x_0 - y\| \leq \|x_0 - c_{Q_K}\| + \|c_{Q_K} - c_K\| \leq 10r_{Q_K} + r_{Q_K} + r_K.$$

(Recall that c_K denotes the center of K and $r_K = \frac{1}{2} \text{diam } K$.) Hence, by (4.45) and (4.48),

$$\|x_0 - y\| \leq 40r_K + 4r_K + r_K = 45r_K \leq \gamma_1 R_1$$

with some absolute constant γ_1 . This proves that

$$(4.49) \quad K \subset Q(x_0, \gamma_1 R_1).$$

Since $\mathcal{I}_L^{(1)} \subset W_E$, the cubes of the family $\mathcal{I}_L^{(1)}$ are non-overlapping. By (4.49), all these cubes are contained in the cube $\tilde{Q}_1 := \gamma_1 Q(x_0, R_1)$, and, by (4.48), the diameter of each such a cube is equivalent to $\text{diam } \tilde{Q}_1 \sim R_1$. This proves that the number of cubes in $\mathcal{I}_L^{(1)}$ does not exceed a constant depending only on n .

In the same way we prove that $\#\mathcal{I}_L^{(2)} \leq \gamma(n)$. In fact, we put $R_2 := \text{dist}(V_L, E \setminus V_L)$ so that $\text{diam } K \sim R_2$ for all $K \in \mathcal{I}_L^{(2)}$. Using the same approach we show that every cube $K \in \mathcal{I}_L^{(2)}$ is contained in a cube $\tilde{Q}_2 := \gamma_2 Q(x_0, R_2)$ where γ_2 is an absolute constant. This implies the required inequality $\#\mathcal{I}_L^{(2)} \leq \gamma(n)$.

The proposition is proved. ■

5. “Projections” of lacunae and interior bridges.

5.1. “Projections” of the lacunae. In this subsection we construct a mapping $\mathcal{L}_E \ni L \mapsto \mathcal{PR}(L) \in E$ which we have mentioned in Section 2. Let us recall its main properties:

- (i) For each lacuna L the point $\mathcal{PR}(L)$ lies in a fixed dilation of the minimal cube of the lacuna, and
- (ii) every point $A \in E$ has at most $C(n)$ “sources”, i.e., lacunae $L' \in \mathcal{L}_E$ such that $\mathcal{PR}(L') = A$.

We refer to the mapping \mathcal{PR} as a “projection” of \mathcal{L}_E into the set E .

The existence of the mapping \mathcal{PR} is proven in Proposition 5.3. This result relies on Lemma 5.1 below. For its formulation we need the following notions: Let S be a closed subset of \mathbf{R}^n , and let $\varepsilon > 0$. As usual, a set $A \subset S$ is said to be an ε -net in S if for each $x \in S$ there exists a point $a_x \in A$ such that $\|a_x - x\| \leq \varepsilon$. We say that points $x, y \in A$ are ε -separated if $\|x - y\| \geq \varepsilon$.

Lemma 5.1 *For every closed set $S \subset \mathbf{R}^n$ there exists a decreasing sequence of non-empty closed sets $\{S_i\}_{i \in \mathbb{Z}}$, $S_{i+1} \subset S_i \subset S$, $i \in \mathbb{Z}$, such that for every $i \in \mathbb{Z}$ the following conditions are satisfied:*

- (i). *The points of the set S_i are 2^i -separated;*
- (ii). *S_i is a 2^{i+1} -net in S .*

Proof. First let us construct the sets S_i for $i \geq 0$.

We let S_0 denote a *maximal* 1-net in S ; thus S_0 is a 1-net in S whose points are 1-separable.

By S_1 we denote a maximal 2-net in S_0 so that S_1 is a 2-net in S_0 whose points are 2-separable. We continue this procedure and at the m -th step we have subsets

$$S \supset S_0 \supset S_1 \supset S_2 \supset \dots \supset S_m$$

such that each set S_i is a 2^i -net in S_{i-1} , and the points of S_i are 2^i -separable. We let S_{m+1} denote a maximal 2^{m+1} -net in S_m so that S_{m+1} is a 2^{m+1} -net in S_m , and the points of S_{m+1} are 2^{m+1} -separable.

Prove that the set S_i is a 2^{i+1} -net in S for every integer $i \geq 0$. In fact, since S_0 is a 1-net in S , for every $x \in S$ there exists $x_0 \in S_0$ such that $\|x - x_0\| \leq 1$. Since S_1 is a 2-net in S_0 , there exists $x_1 \in S_1$ such that $\|x_0 - x_1\| \leq 2$. Continuing this process we obtain points $x_j \in S_j$, $j = 0, 1, \dots, i$, such that $\|x_j - x_{j+1}\| \leq 2^{j+1}$, $j = 0, 1, \dots, i-1$.

Hence

$$\|x - x_i\| \leq \|x - x_0\| + \sum_{j=0}^{i-1} \|x_j - x_{j+1}\| \leq \sum_{j=0}^i 2^j = 2^{i+1} - 1 \leq 2^{i+1}$$

proving that S_i is a 2^{i+1} -net in S whenever $i \geq 0$.

Let us construct sets S_i for $i < 0$. To this end we let S_{-1} denote a *maximal* 2^{-1} -net in S containing S_0 . Then the set S_{-1} is a 2^{-1} -net in S and its points are 2^{-1} -separable. At the same way we construct a set S_{-2} as a maximal 2^{-2} -net in S containing S_1 . We continue this inductive procedure and in this way we obtain the required sequence of sets $\{S_i\}$ whenever $i < 0$.

The lemma is proved. ■

Proposition 5.2 *Let S be a closed subset of \mathbf{R}^n and let \mathcal{Q} be a family of non-overlapping cubes in \mathbf{R}^n . Suppose that there exist constants $\eta, \theta \geq 1$ such that for every cube $Q \in \mathcal{Q}$ the following inequality*

$$(5.1) \quad \text{diam } Q \leq \eta \text{diam}(\theta Q \cap S)$$

holds.

Then there exists a mapping $\mathcal{Q} \ni Q \mapsto w_Q \in S$ such that

(i). $w_Q \in (2\theta)Q \cap S$ for every $Q \in \mathcal{Q}$;

(ii). for every $a \in S$ we have

$$\#\{Q \in \mathcal{Q} : w_Q = a\} \leq \gamma.$$

Here γ is a constant depending only on n, η and θ .

Proof. Let $Q \in \mathcal{Q}$ and let $i_Q \in \mathbb{Z}$ be such an integer that

$$(5.2) \quad 2^{i_Q} < \text{diam}(\theta Q \cap S) \leq 2^{i_Q+1}.$$

Let $\{S_i\}_{i \in \mathbb{Z}}$ be a family of subsets of S satisfying conditions of Lemma 5.1. Let a_Q, b_Q be points on S such that $a_Q, b_Q \in (\theta Q) \cap S$ and

$$\|a_Q - b_Q\| = \text{diam}(\theta Q \cap S).$$

Then, by (5.2),

$$(5.3) \quad 2^{i_Q} < \|a_Q - b_Q\| \leq 2^{i_Q+1}.$$

By Lemma 5.1, the set S_{i_Q-2} is a 2^{i_Q-1} -net in S , so that there exist points $A_Q, B_Q \in S_{i_Q-2}$ such that

$$(5.4) \quad \|A_Q - a_Q\| \leq 2^{i_Q-1} \quad \text{and} \quad \|B_Q - b_Q\| \leq 2^{i_Q-1}.$$

Prove that $A_Q \neq B_Q$ and

$$(5.5) \quad \{A_Q, B_Q\} \cap (S_{i_Q-2} \setminus S_{i_Q+2}) \neq \emptyset.$$

In fact, by (5.4),

$$\|a_Q - b_Q\| \leq \|a_Q - A_Q\| + \|A_Q - B_Q\| + \|B_Q - b_Q\| \leq 2^{i_Q-1} + \|A_Q - B_Q\| + 2^{i_Q-1}$$

so that

$$\|A_Q - B_Q\| \geq \|a_Q - b_Q\| - 2^{i_Q}.$$

By (5.2), $\|a_Q - b_Q\| > 2^{i_Q}$ so that $\|A_Q - B_Q\| > 0$ proving that $A_Q \neq B_Q$.

Prove that $\{A_Q, B_Q\} \not\subset S_{i_Q+2}$. If $\{A_Q, B_Q\} \subset S_{i_Q+2}$, then, by property (i) of Lemma 5.1, A_Q and B_Q are 2^{i_Q+2} -separable so that $\|A_Q - B_Q\| \geq 2^{i_Q+2}$. On the other hand, by (5.3) and (5.4),

$$\|A_Q - B_Q\| \leq \|A_Q - a_Q\| + \|a_Q - b_Q\| + \|b_Q - B_Q\| \leq 2^{i_Q-1} + 2^{i_Q+1} + 2^{i_Q-1} = 3 \cdot 2^{i_Q} < 2^{i_Q+2},$$

a contradiction. Since $A_Q, B_Q \in S_{i_Q-2}$, the property (5.5) follows.

Now we are in a position to define a mapping $w : \mathcal{Q} \rightarrow S$ satisfying conditions (i) and (ii) of the proposition. Given a cube $Q \in \mathcal{Q}$ by w_Q we denote a point from the set $\{A_Q, B_Q\}$ which belongs to the set $S_{i_Q-2} \setminus S_{i_Q+2}$ (thus either $w_Q = A_Q$ or $w_Q = B_Q$). By (5.5), such a point exists.

Thus $w_Q \in S_{i_Q-2} \setminus S_{i_Q+2}$. Prove that $w_Q \in (2\theta)Q \cap S$, i.e., condition (i) of the proposition is satisfied. In fact, by (5.2),

$$2^{i_Q} \leq \text{diam}(\theta Q \cap S) \leq \text{diam}(\theta Q) = \theta \text{diam } Q.$$

Combining this inequality with (5.4), we obtain

$$\|A_Q - a_Q\| \leq 2^{i_Q-1} \leq \frac{1}{2}\theta \text{diam } Q.$$

Since $a_Q \in \theta Q$, this implies that $A_Q \in (2\theta)Q$. In the same way we prove that $B_Q \in (2\theta)Q$.

Since w_Q coincides either with A_Q or with B_Q , we conclude that $w_Q \in (2\theta)Q$ as well. Since $w_Q \in S$, the property (i) of the proposition follows.

Prove the property (ii) of the proposition. Let $a \in S$. Suppose that $Q, K \in \mathcal{Q}$ and $w_Q = w_K = a$. Recall that

$$a = w_Q \in S_{i_Q-2} \setminus S_{i_Q+2} \quad \text{and} \quad a = w_K \in S_{i_K-2} \setminus S_{i_K+2}.$$

Since $a \notin S_{i_Q+2}$, $a \in S_{i_K-2}$ and $\{S_i\}$ is a decreasing sequence of sets, we conclude that

$$i_K - 2 < i_Q + 2$$

so that $i_K < i_Q + 4$. In the same fashion we prove that $i_Q < i_K + 4$ so that

$$(5.6) \quad i_Q - 4 < i_K < i_Q + 4.$$

But, by (5.2) and (5.1),

$$\text{diam } Q \leq \eta \text{diam}(\theta Q \cap S) \leq \eta 2^{i_Q+1}$$

and

$$2^{i_Q} \leq \text{diam}(\theta Q \cap S) \leq \theta \text{diam } Q.$$

Hence

$$(2\eta)^{-1} \text{diam } Q \leq 2^{i_Q} \leq \theta \text{diam } Q.$$

The same is true for the cube K , i.e.,

$$(2\eta)^{-1} \text{diam } K \leq 2^{i_K} \leq \theta \text{diam } K.$$

Combining these inequalities with (5.6), we obtain

$$(2\eta)^{-1} \text{diam } K \leq 2^{i_K} \leq 16 \cdot \theta \text{diam } Q$$

proving that

$$(5.7) \quad \text{diam } K \leq \gamma_1 \text{diam } Q$$

with $\gamma_1 := 32\eta\theta$. In the same way we show that

$$(5.8) \quad \text{diam } Q \leq \gamma_1 \text{diam } K.$$

Let

$$\mathcal{Q}_a := \{Q \in \mathcal{Q} : w_Q = a\}.$$

By (5.7) and (5.8), $|Q'| \sim |Q''|$ for every $Q', Q'' \in \mathcal{Q}_a$ and constants of this equivalence depend only on η and θ . Furthermore, $(2\theta)Q \ni a$ for each $Q \in \mathcal{Q}_a$. Since the cubes of the family \mathcal{Q}_a are non-overlapping, we conclude that $\#\mathcal{Q}_a \leq C(n, \eta, \theta)$.

The proposition is completely proved. ■

Proposition 5.3 *There exist an absolute constant $\gamma > 0$ and a mapping*

$$\mathcal{L}_E \ni L \longrightarrow \mathcal{PR}(L) \in E$$

such that:

(i). *For every lacuna $L \in \mathcal{L}_E$ we have*

$$\mathcal{PR}(L) \in (\gamma Q_L) \cap E ;$$

(ii). *For every $a \in E$*

$$\#\{L \in \mathcal{L}_E : \mathcal{PR}(L) = a\} \leq C(n).$$

Proof. We let $\mathcal{Q}_E := \{Q_L : L \in \mathcal{L}_E\}$ denote the family of all “generalized” cubes Q_L of the set E . We use the word “generalized” to emphasize that if $\text{diam } Q_L > 0$, then $Q_L \in W_E$ is a Whitney cube, while Q_L is a point of E provided $\text{diam } Q_L = 0$. Thus \mathcal{Q}_E is a subset of the set $W_E \cup E$.

We put

$$(5.9) \quad \mathcal{PR}(L) := Q_L \quad \text{if} \quad \text{diam } Q_L = 0.$$

Let

$$\tilde{\mathcal{Q}}_E := \{Q_L : L \in \mathcal{L}_E, \text{diam } Q_L > 0\}.$$

Since $\tilde{\mathcal{Q}}_E \subset W_E$, this family consists of non-overlapping cubes. By Corollary 4.6, every cube $Q \in \tilde{\mathcal{Q}}_E$ satisfies inequality (5.1) with $\theta = 90$. Hence, by Proposition 5.2, there exists a mapping

$$\tilde{\mathcal{Q}}_E \ni Q \longrightarrow w_Q \in E$$

such that

$$w_Q \in (\gamma Q) \cap E, \quad Q \in \tilde{\mathcal{Q}}_E,$$

with $\gamma = 2\theta = 180$. Furthermore, for every $a \in E$

$$\#\{Q \in \tilde{\mathcal{Q}}_E : w_Q = a\} \leq C(n).$$

We define $\mathcal{PR}(L)$ by letting $\mathcal{PR}(L) := w_{Q_L}$. Then, by the latter inequality and (5.9),

$$\#\{L \in \mathcal{L}_E : \mathcal{PR}(L) = a\} \leq C(n) + 1,$$

and the proposition follows. ■

Corollary 5.4 *Let E be a finite subset of \mathbf{R}^n . Then the number of its lacunae does not exceed $C(n)\#E$.*

Proof. By part (ii) of Proposition 5.3, for each $a \in E$ we have

$$\#\{L \in \mathcal{L}_E : \mathcal{PR}(L) = a\} \leq C(n).$$

Hence

$$\#\mathcal{L}_E \leq \sum_{a \in E} \#\{L \in \mathcal{L}_E : \mathcal{PR}(L) = a\} \leq C(n)\#E. \quad \blacksquare$$

5.2. A graph of lacunae and interior bridges. Let $L \in \mathcal{L}_E$ be a lacuna. Recall that

$$U_L = \cup\{Q : Q \in L\}.$$

Definition 5.5 Let $L, L' \in \mathcal{L}_E$ be lacunae. We say that L and L' are *contacting lacunae* if $U_L \cap U_{L'} \neq \emptyset$. In this case we write $L \leftrightarrow L'$.

Thus $L \leftrightarrow L'$ whenever there exist cubes $Q \in L$ and $Q' \in L'$ such that $Q \cap Q' \neq \emptyset$. We refer to the pair of such cubes as *contacting cubes*. Let us present several properties of contacting lacunae and contacting cubes which directly follow from results of the previous subsections.

Proposition 5.6 (i). *Every lacuna $L \in \mathcal{L}_E$ contacts with at most $C(n)$ lacunae, i.e.,*

$$\#\{L' \in \mathcal{L}_E : L' \leftrightarrow L\} \leq C(n);$$

(ii). *Every true lacuna contacts only with elementary lacunae.*

Clearly, part (i) of the proposition follows from Proposition 4.11, and part (ii) follows from part (i) of Proposition 4.10.

In turn, part (ii) of Proposition 4.10 imply the following

Proposition 5.7 *Let $L \in \mathcal{L}_E$ be a lacuna and let $Q \in L$ be a contacting cube. (I.e., there exist a lacuna $L' \in \mathcal{L}_E$ and a cube $Q' \in L'$ such that $Q \cap Q' \neq \emptyset$.) Then either*

$$\text{diam } Q \sim \text{diam } V_L \sim \min\{\text{diam } K : K \in L\} = \text{diam } Q_L$$

or

$$\text{diam } Q \sim \text{dist}(V_L, E \setminus V_L) \sim \max\{\text{diam } K : K \in L\} = \text{diam } Q^{(L)}$$

with absolute constants in the equivalences.

The relation \leftrightarrow determines a certain graph structure on \mathcal{L}_E . We denote this graph by \mathcal{GL}_E . Thus two vertices of this graph, i.e., two lacunae $L, L' \in \mathcal{L}_E$, are joined by an edge if they are contacting lacunae \iff there exist Whitney cubes $Q \in L$ and $Q' \in L'$ such that $Q \cap Q' \neq \emptyset$.

By part (i) of Proposition 5.6, the degree of every vertex of the graph \mathcal{GL}_E is bounded by a constant $C = C(n)$. In turn, by Corollary 5.4, for every finite set $E \subset \mathbf{R}^n$ the number of vertices of \mathcal{GL}_E does not exceed the number of points of E (up to a multiplicative constant $C = C(n)$).

We turn to definition of an interior bridge of a lacuna.

Definition 5.8 Let $L \in \mathcal{L}_E$ be a lacuna. We define two points $A_L, B_L \in E$ as follows: If $\text{diam } V_L > 0$, then we choose $A_L, B_L \in V_L$ to be any pair of distinct points in V_L such that

$$\|A_L - B_L\| = \text{diam } V_L.$$

If $\text{diam } V_L = 0$, i.e., V_L is a single point, we set $A_L = V_L$. We choose B_L to be some point in $E \setminus \{A_L\}$ whose distance from A_L is minimal.

We refer to the ordered couple $T(L) = (A_L, B_L)$ as an *interior bridge* of the lacuna L .

Given lacunae $L, L' \in \mathcal{L}_E$ we put

$$d(L, L') := \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\}.$$

Proposition 5.9 Let $L, L' \in \mathcal{L}_E$ be contacting lacunae and let $Q \in L$ and $Q' \in L'$ be contacting cubes (i.e., $Q \cap Q' \neq \emptyset$). Then

$$d(L, L') \sim \text{diam } Q \sim \text{diam } Q'$$

with absolute constants in the equivalences. Furthermore,

$$\{A_L, B_L, A_{L'}, B_{L'}\} \subset (\gamma Q) \cap (\gamma Q')$$

where $\gamma > 0$ is an absolute constant.

Proof. By Lemma 4.2, $\text{diam } Q \sim \text{diam } Q'$. Since $Q \cap Q' \neq \emptyset$, for some absolute constant $\gamma_1 > 0$ we have

$$(5.10) \quad Q \subset \gamma_1 Q' \quad \text{and} \quad Q' \subset \gamma_1 Q.$$

Recall that $V_L = (90K) \cap E$ for each cube $K \in L$, see (4.5), so that

$$V_L \subset 90Q \quad \text{and} \quad V_{L'} \subset 90Q'.$$

Combining this with (5.10) we obtain

$$(5.11) \quad V_L \cup V_{L'} \subset (\gamma_2 Q) \cap (\gamma_2 Q').$$

Prove that

$$(5.12) \quad \{A_L, B_L, A_{L'}, B_{L'}\} \subset (\gamma_3 Q) \cap (\gamma_3 Q').$$

First we note that, by Proposition 4.10, either L or L' is an elementary lacuna. Suppose that L' is such a lacuna, i.e., $L' \in \hat{\mathcal{L}}$. Then $L' = \{Q'\}$ and, by (4.7),

$$\text{diam } V_{L'} = \text{diam}((90Q') \cap E) \sim \text{diam } Q'.$$

Since $\text{diam } Q' > 0$, by Definition 5.8, $A_{L'}, B_{L'} \in V_{L'}$ and

$$\|A_{L'} - B_{L'}\| = \text{diam } V_{L'}$$

so that

$$(5.13) \quad \|A_{L'} - B_{L'}\| \sim \text{diam } Q'.$$

Now let us consider the lacuna L . If $\text{diam } V_L > 0$, then again, by Definition 5.8, $A_L, B_L \in V_L$ so that, by (5.11), the inclusion (5.12) is satisfied.

Suppose that $\text{diam } V_L = 0$, i.e., $V_L = \{A_L\}$. In this case, by Definition 5.8, B_L is a point nearest to A_L on the set $E \setminus \{A_L\}$. Thus

$$(5.14) \quad \|A_L - B_L\| = \text{dist}(A_L, E \setminus \{A_L\}) = \text{dist}(V_L, E \setminus V_L).$$

Recall that Q is a contacting cube so that, by Proposition 5.7, either $\text{diam } Q \sim \text{diam } V_L$ or

$$(5.15) \quad \text{diam } Q \sim \text{dist}(V_L, E \setminus V_L).$$

Since $\text{diam } V_L = 0$, we conclude that equivalence (5.15) holds. Combining this with equality (5.14), we obtain $\|A_L - B_L\| \sim \text{diam } Q$. Since $A_L \subset 90Q$, this implies that $B_L \subset \gamma_3 Q$ for a certain absolute constant $\gamma_3 \geq \gamma_2$. This proves inclusion (5.12).

In particular, by this inclusion,

$$d(L, L') := \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\} \leq \gamma_3 \text{diam } Q$$

and $d(L, L') \leq \gamma_3 \text{diam } Q'$.

On the other hand, by (5.13),

$$\text{diam } Q' \sim \|A_{L'} - B_{L'}\| \leq \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\} = d(L, L')$$

proving that $d(L, L') \sim \text{diam } Q \sim \text{diam } Q'$.

The proof of the proposition is complete. ■

6. Bridges between lacunae.

In this section we study *bridges between contacting lacunae* which we have briefly described in Section 2. This important notion provides a useful tool for the study of a lacunary modification of the Whitney extension method which we introduce in the next section.

We define bridges in several steps. First of all let us slightly generalize the notion of the “triangle” introduced in the first section. We have defined a triangle as a subset $\Delta = \{z_1, z_2, z_3\} \subset \mathbf{R}^2$ consisting of three non-collinear points. In what follows we refer to such a subset as a “*true*” triangle.

Let us also consider a subset $\Delta = \{z_1, z_2, z_3\} \subset \mathbf{R}^2$ consisting of *three collinear points*. In this case we call Δ a “*degenerate*” triangle. Thus a degenerate triangle is the “triangle” formed by three collinear points.

We refer to the points z_1, z_2, z_3 as the vertices of the triangle $\Delta = \{z_1, z_2, z_3\}$. Also given three points $A, B, C \in \mathbf{R}^2$ we let $\Delta\{A, B, C\}$ denote the triangle with vertices in these points.

We say that a side $[A, B]$ is *the smallest side* of the triangle $\Delta\{A, B, C\}$ if

$$\|A - B\| < \min\{\|A - C\|, \|B - C\|\}.$$

Definition 6.1 Let $L, L' \in \mathcal{L}_E$ be contacting lacunae such that

$$(6.1) \quad \{A_L, B_L\} \cap \{A_{L'}, B_{L'}\} = \emptyset.$$

We say that the (non-ordered) pair

$$T(L, L') = \{C(L, L'), C(L', L)\}$$

where $C(L, L') \in \{A_L, B_L\}$ and $C(L', L) \in \{A_{L'}, B_{L'}\}$, is an *exterior bridge* between lacunae L and L' if the following conditions are satisfied:

In the triangle $\Delta\{A_L, B_L, C(L', L)\}$ the side of the triangle which is opposite to the vertex $C(L, L')$ is not the smallest side of the triangle.

A similar condition holds for the triangle $\Delta\{A_{L'}, B_{L'}, C(L, L')\}$: the side of the triangle which is opposite to the vertex $C(L', L)$ is not the smallest side of the triangle.

We also say that the bridge $T(L, L')$ is *connected* to the bridge $T(L)$ and to the bridge $T(L')$.

Remark 6.2 In the sequel we identify the exterior bridges $T(L, L')$ and $T(L', L)$ between contacting lacunae L and L' ; thus $T(L, L') = T(L', L)$ for every $L, L' \in \mathcal{L}_E$ such that $L \leftrightarrow L'$. \triangleleft

Proposition 6.3 Let $L, L' \in \mathcal{L}_E$ be contacting lacunae satisfying condition (6.1). Then there exists an exterior bridge $T(L, L') = \{C(L, L'), C(L', L)\}$ between these lacunae. Furthermore,

$$\text{diam } \Delta\{A_L, B_L, C(L', L)\} \sim \text{diam } \Delta\{A_{L'}, B_{L'}, C(L, L')\} \sim \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\}$$

with absolute constants in the equivalences.

Proof. Consider three cases.

The first case. Suppose that

$$(6.2) \quad \|A_L - B_L\| = \min\{\|a - b\| : a, b \in \{A_L, B_L, A_{L'}, B_{L'}\}, a \neq b\}.$$

We put $C(L, L') = A_L$. Let $C(L', L) \in \{A_{L'}, B_{L'}\}$ be such a point that

$$(6.3) \quad \|C(L', L) - A_L\| = \min\{\|A_L - A_{L'}\|, \|A_L - B_{L'}\|\}.$$

Prove that the couple

$$(6.4) \quad T(L, L') = \{C(L, L'), C(L', L)\}$$

provides an exterior bridge between the lacunae L and L' .

In fact, by (6.2), the side $[A_L, B_L]$ is a minimal side of the triangle $\Delta\{A_L, B_L, C(L', L)\}$ so that the side of this triangle which is opposite to the vertex $C(L, L') = A_L$ is not the smallest side.

On the other hand, by equality (6.3), the side of the triangle $\Delta\{A_{L'}, B_{L'}, C(L, L')\}$ which is opposite to the vertex $C(L', L)$ is not the smallest because it is not smaller than the side $[C(L, L'), C(L', L)] = [A_L, C(L', L)]$.

These observations shows that the couple (6.4) satisfies the conditions of Definition 6.1 providing an exterior bridge between these lacunae.

The second case. Let

$$(6.5) \quad \|A_{L'} - B_{L'}\| = \min\{\|a - b\| : a, b \in \{A_L, B_L, A_{L'}, B_{L'}\}, a \neq b\}.$$

We prove the existence of the corresponding bridge in the same fashion as in the previous case.

The third case. We assume that both condition (6.2) and condition (6.5) are not satisfied. In this case there exist points $C(L, L') \in \{A_L, B_L\}$ and $C(L', L) \in \{A_{L'}, B_{L'}\}$ such that

$$(6.6) \quad \|C(L, L') - C(L', L)\| = \min\{\|a - b\| : a, b \in \{A_L, B_L, A_{L'}, B_{L'}\}, a \neq b\}.$$

It can be readily seen that in this case the couple (6.4) is a an exterior bridge between L and L' . In fact, consider the triangle $\Delta = \Delta\{A_L, B_L, C(L', L)\}$ and the vertex $C(L, L') \in \{A_L, B_L\}$. By (6.6), the side $[C(L, L'), C(L', L)]$ is a side of minimal length in Δ . Therefore the side of Δ which is opposite to $C(L, L')$ is not the smallest side in this triangle.

In the same way we prove that the side of the triangle $\Delta' = \Delta\{A_{L'}, B_{L'}, C(L, L')\}$ which is opposite to the vertex $C(L', L)$ is not the smallest side of Δ' . The proof of the proposition is finished. \blacksquare

Let us consider two contacting lacunae $L, L' \in \mathcal{L}_E$, $L \leftrightarrow L'$, and their interior bridges $T(L) = (A_L, B_L)$ and $T(L') = (A_{L'}, B_{L'})$. Suppose that $\{A_L, B_L\} \neq \{A_{L'}, B_{L'}\}$ but

$$(6.7) \quad \{A_L, B_L\} \cap \{A_{L'}, B_{L'}\} \neq \emptyset.$$

In other words two point sets $\{A_L, B_L\}$ and $\{A_{L'}, B_{L'}\}$ are different and have a unique common point. We denote this point by $D(L, L')$. By $E(L)$ we denote the remaining point of the set $\{A_L, B_L\}$; thus $\{A_L, B_L\} = \{D(L, L'), E(L)\}$. In a similar way we define a point $E(L')$; thus $\{A_{L'}, B_{L'}\} = \{D(L, L'), E(L')\}$.

Let us define external bridges for this case. Consider two cases.

The first case: The side $[E(L), E(L')]$ is *the smallest* side of a triangle Δ with vertices in $E(L), E(L')$, and $D(L, L')$. In this case we put

$$C(L, L') = E(L), \quad C(L', L) = E(L')$$

and define an exterior bridge $T(L, L') = \{C(L, L'), C(L', L)\}$ between the lacunae L and L' . Note that the bridge $T(L, L')$ satisfies the conditions of Definition 6.1.

As before we say that *the bridge $T(L, L')$ is connected to the bridge $T(L)$ and to the bridge $T(L')$.*

The second case: The side $[E(L), E(L')]$ is not *the smallest* side of the triangle Δ . In this case *we do not introduce any additional bridges between L and L' . We only say that the (interior) bridges $T(L)$ and $T(L')$ are connected.*

It remains to consider the last case of a pair of contacting lacunae $L, L' \in \mathcal{L}_E$, $L \leftrightarrow L'$, such that $\{A_L, B_L\} = \{A_{L'}, B_{L'}\}$. Similar to the previous case we do not introduce any

additional bridges between L and L' , and refer to (interior) bridges $T(L)$ and $T(L')$ as *connected bridges*.

We have defined all types of bridges (interior and exterior). We have also introduced the notion of connected bridges.

By \mathcal{BR}_E we denote the family of all bridges constructed for the set E . Given a bridge $T \in \mathcal{BR}_E$ by $A^{[T]}$ and $B^{[T]}$ we denote its ends. Note that $A^{[T]} \neq B^{[T]}$ for every bridge $T \in \mathcal{BR}_E$. In particular, if $T = T(L)$ is an interior bridge of a lacuna $L \in \mathcal{L}_E$, these points coincide with the points A_L and B_L , see Definition 5.8.

We have also introduced the notion of *connected bridges*. We denote connected bridges by the sign \longleftrightarrow .

Let us note several general properties of bridges.

- If two bridges T and T' are connected ($T \longleftrightarrow T'$), then *one of them is an interior bridge*.

- Interior bridges $T, T' \in \mathcal{BR}_E$ are connected ($T \longleftrightarrow T'$) provided they have the same ends, i.e., $\{A^{[T]}, B^{[T]}\} = \{A^{[T']}, B^{[T']}\}$.

- If the ends of two connected bridges T and T' are different ($(A^{[T]}, B^{[T]}) \neq (A^{[T']}, B^{[T']})$) the bridges have a unique common end which we denote by $D = D(T, T')$. Thus in this case

$$\# \{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} = 3$$

so that these points form a triangle Δ . This triangle possesses the following property: *the side of Δ which is opposite to the vertex D is not the smallest side of this triangle*.

In turn, this property implies the following one: Let $\alpha(D)$ be the angle corresponding to the vertex D . Then

$$(6.8) \quad |\sin \alpha(D)| \sim \frac{1}{R_\Delta} \text{diam } \Delta = \mathbf{c}_\Delta \text{diam } \Delta$$

with absolute constants in the equivalences. In fact, using a formula from the elementary geometry, we obtain

$$|\sin \alpha(D)| = \frac{1}{2} \mathbf{c}_\Delta \ell$$

where ℓ is the Euclidean length of the side opposite to the vertex D . Since this side is not the smallest in Δ , we have $\text{diam } \Delta \sim \ell$ proving (6.8).

Finally, by \mathcal{GB}_E we denote a graph whose vertices is the family \mathcal{BR}_E of all bridges (interior and exterior). In this graph two vertices (bridges) T and T' are joined by an edge if $T \longleftrightarrow T'$, i.e., the bridges T and T' are connected. This graph possesses the following property: Let $L, L' \in \mathcal{L}_E, L \leftrightarrow L'$, be contacting lacunae. Then either their interior bridges $T(L)$ and $T(L')$ are connected, or there exists an (exterior) bridge which is connected to both the bridge $T(L)$ and to the bridge $T(L')$.

The next proposition enables us to every couple of connected bridges to assign a certain Whitney cube which is “close” to the set of ends of these bridges.

Proposition 6.4 *Let $T, T' \in \mathcal{BR}_E, T \longleftrightarrow T'$, be a pair of connected bridges with the ends at the points $\{A^{[T]}, B^{[T]}\}$ and $\{A^{[T']}, B^{[T']}\}$. Let the bridge T be the interior bridge of a*

lacuna $L \in \mathcal{L}_E$. Then there exists a cube $\widehat{Q}(T, T') \in W_E$ which coincides either with the cube Q_L or with the cube $Q^{(L)}$ such that

$$(6.9) \quad \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \sim \text{diam } \widehat{Q}(T, T')$$

and

$$(6.10) \quad \{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \subset \gamma \widehat{Q}(T, T').$$

Here the constant γ and the constants of the equivalence in (6.9) are absolute.

Proof. Since T is the interior bridge of the lacuna L , we have $A^{[T]} = A_L, B^{[T]} = B_L$. Consider two case:

The first case. Suppose that the bridge T' is the interior bridge of a lacuna $L' \in \mathcal{L}_E$ which contacts to L ($L \leftrightarrow L'$). In this case $A^{[T']} = A_{L'}, B^{[T']} = B_{L'}$ and the sets of the ends $\{A_L, B_L\}$ and $\{A_{L'}, B_{L'}\}$ have a common point.

Let $Q \in L$ and $Q' \in L'$ be contacting cubes. Then, by Proposition 5.7, either

$$(6.11) \quad \text{diam } Q \sim \text{diam } Q_L,$$

or

$$\text{diam } Q \sim \text{diam } Q^{(L)}$$

with absolute constants in these equivalences.

On the other hand, by Proposition 5.9,

$$(6.12) \quad \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \sim \text{diam } Q$$

and

$$(6.13) \quad \{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \subset \gamma_1 Q$$

where γ_1 is an absolute constant.

Now we put $\widehat{Q}(T, T') = Q_L$ if (6.11) is satisfied; otherwise we put $\widehat{Q}(T, T') = Q^{(L)}$. In particular, $\widehat{Q}(T, T') \in L$.

In both cases

$$(6.14) \quad \text{diam } Q \sim \text{diam } \widehat{Q}(T, T').$$

Combining this equivalence with (6.12) we conclude that (6.9) holds.

Prove (6.10). Since $Q, \widehat{Q}(T, T') \in L$,

$$V_L = (90Q) \cap E = (90\widehat{Q}(T, T')) \cap E$$

so that

$$(90Q) \cap (90\widehat{Q}(T, T')) \neq \emptyset.$$

This property and (6.14) imply the inclusion $Q \subset \gamma_2 \widehat{Q}(T, T')$ with a certain absolute constant γ_2 . Combining this inclusion with (6.13) we obtain (6.10) proving the proposition in the case under consideration.

The second case. Let T' be an exterior bridge which connects the lacuna L to a contacting lacuna $L' \in \mathcal{L}_E$. Let $Q \in L, Q' \in L'$ be corresponding contacting cubes (thus $Q \cap Q' \neq \emptyset$). In this case the ends of the bridge T' can be identified with the points $C(L, L') \in \{A_L, B_L\}$ and $C(L', L) \in \{A_{L'}, B_{L'}\}$. Thus

$$\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} = \{A_L, B_L, C(L', L)\}.$$

By Definition 5.8, if $\text{diam } V_L > 0$, then

$$(6.15) \quad A_L, B_L \in V_L \quad \text{and} \quad \|A_L - B_L\| = \text{diam } V_L.$$

If $\text{diam } V_L = 0$, then

$$(6.16) \quad V_L = \{A_L\}, \quad B_L \in E \setminus V_L,$$

and

$$(6.17) \quad \|A_L - B_L\| = \text{dist}(V_L, E \setminus V_L).$$

Suppose that $\text{diam } V_L > 0$ and $C(L', L) \in V_L$. In this case

$$\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} = \{A_L, B_L, C(L', L)\} \subset V_L$$

so that, by (6.15),

$$\text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} = \text{diam}\{A_L, B_L, C(L', L)\} = \text{diam } V_L \sim \text{diam } Q_L.$$

Furthermore, since $Q_L \in L$, we have $V_L = (90Q_L) \cap E$ proving that $V_L \subset 90Q_L$. Thus in this case we can put $\widehat{Q}(T, T') := Q_L$.

Now suppose that $\text{diam } V_L > 0$ and $C(L', L) \in E \setminus V_L$. Then

$$(6.18) \quad \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} = \text{diam}\{A_L, B_L, C(L', L)\} \geq \text{dist}(V_L, E \setminus V_L).$$

By Proposition 4.9,

$$\text{diam } Q^{(L)} \leq \gamma \text{dist}(V_L, E \setminus V_L)$$

so that

$$\text{diam } Q^{(L)} \leq \gamma \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}.$$

On the other hand, by Proposition 5.9,

$$\begin{aligned} \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} &= \text{diam}\{A_L, B_L, C(L', L)\} \leq \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\} \\ &\leq \gamma \text{diam } Q \leq \gamma \text{diam } Q^{(L)}. \end{aligned}$$

(Recall that the cube $Q^{(L)}$ has the maximal diameter in the lacuna L .) Hence

$$(6.19) \quad \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \sim \text{diam } Q^{(L)}.$$

Furthermore, by Proposition 5.9,

$$\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \subset \{A_L, B_L, A_{L'}, B_{L'}\} \subset \gamma_1 Q.$$

Since $Q, Q^{(L)} \in L$, we have $(90Q) \cap (90Q^{(L)}) \neq \emptyset$. But $\text{diam } Q \leq \text{diam } Q^{(L)}$ so that $Q \subset \gamma_2 Q^{(L)}$ for some absolute constant γ_2 . Hence

$$(6.20) \quad \{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \subset \gamma_3 Q^{(L)}.$$

This enables us to put $\widehat{Q}(T, T') := Q^{(L)}$.

Consider the remaining case $\text{diam } V_L = 0$, see (6.16) and (6.17). Since

$$\|A_L - B_L\| = \text{dist}(V_L, E \setminus V_L),$$

inequality (6.18) is satisfied. Now repeating the considerations of the previous case we show that equivalence (6.19) and inclusion (6.20) hold. This again enables us to put in this case $\widehat{Q}(T, T') := Q^{(L)}$.

The proof of the proposition is complete. ■

Definition 6.5 We say that \mathcal{Q} is a family of well-separated cubes if for every pair of cubes $Q, Q' \in \mathcal{Q}, Q \neq Q'$, the following inequality

$$\text{diam } Q + \text{diam } Q' \leq \text{dist}(Q, Q')$$

holds.

Proposition 6.6 *There exists a family \mathcal{K}_E of well-separated cubes and a one-to-one mapping which to every pair of connected bridges $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$, assigns a cube $K(T, T') \in \mathcal{K}_E$ such that*

$$(6.21) \quad \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \sim \text{diam } K(T, T')$$

and

$$(6.22) \quad \{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \subset \gamma K(T, T').$$

Here the constant γ and the constants of the equivalence in (6.21) depend only on n .

Proof. Let

$$\mathcal{M}_E := \{Q_L, Q^{(L)} : L \in \mathcal{L}_E\}$$

be a family of all “minimal” and “maximal” cubes of lacunae of the set E . Let

$$\mathcal{CB}_E := \{(T, T') : T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'\}$$

be the family of all (non-ordered) pairs of connected bridges. In Proposition 6.4 we have constructed a mapping

$$(6.23) \quad \mathcal{CB}_E \ni (T, T') \mapsto \widehat{Q}(T, T') \in \mathcal{M}_E$$

which to every pair of connected bridges $(T, T') \in \mathcal{CB}_E$ assigns a cube $\widehat{Q}(T, T') \in \mathcal{M}_E$ satisfying conditions (6.9) and (6.10). Of course, by these conditions, this mapping satisfies the conditions (6.21) and (6.22) as well. However in general the mapping (6.23) is not one-to-one mapping so that we can not put $K(T, T') = \widehat{Q}(T, T')$.

Nevertheless the mapping $(T, T') \rightarrow \widehat{Q}(T, T')$ is “almost” one-to-one, i.e., every cube $Q \in \mathcal{M}_E$ has at most $N(n)$ origins. This enables us to obtain the required mapping $(T, T') \rightarrow K(T, T')$ by a slight modification of the mapping (6.23). Namely we fix a family consisting of $N(n)$ equal pairwise disjoint subcubes of the cube Q whose diameters are equivalent to $\text{diam } Q$. Then to each origin of Q we assign in a one-to-one way a subcube from this family.

Following this scheme let us first prove that the mapping $(T, T') \rightarrow \widehat{Q}(T, T')$ is “almost” one-to-one. In fact, let $L \in \mathcal{L}_E$ be a lacuna. Then, by part (i) of Proposition 5.6, there exist at most $C(n)$ lacunae $L' \in \mathcal{L}_E$ which contact to L ($L \leftrightarrow L'$). We recall that L is connected by bridges only to *contacting* lacunae so that the number of bridges connected to the lacuna L is bounded by $C(n)$ as well.

Let $T_L = (A_L, B_L)$ be the interior bridge of the lacuna L . By Proposition 6.4, for every bridge T connected to T_L the cube $Q(T_L, T)$ coincides either with the cube Q_L (i.e., with the “minimal” cube of the lacuna L), or with the cube $Q^{(L)}$ (the “maximal” cube of the lacuna L). This motivates us to introduce two family of cubes: a family

$$\mathcal{A}_L := \{T \in \mathcal{BR}_E : T \longleftrightarrow T_L, Q(T_L, T) = Q_L\}$$

and a family

$$\mathcal{A}^{(L)} := \{T \in \mathcal{BR}_E : T \longleftrightarrow T_L, Q(T_L, T) = Q^{(L)}\}.$$

Clearly, \mathcal{A}_L and $\mathcal{A}^{(L)}$ is a partition of the family

$$\mathcal{A}(L) := \{T \in \mathcal{BR}_E : T \longleftrightarrow T_L\}$$

of all bridges connected to T_L , i.e., $\mathcal{A}(L) = \mathcal{A}_L \cup \mathcal{A}^{(L)}$ and $\mathcal{A}_L \cap \mathcal{A}^{(L)} = \emptyset$. We know that the family $\mathcal{A}(L)$ is finite and $\#\mathcal{A}(L) \leq C(n)$.

Let us define the mapping $(T, T') \rightarrow K(T, T')$ on the pairs (T_L, T) where $T \in \mathcal{A}_L$. Since

$$m_L := \#\mathcal{A}_L \leq C(n)$$

we can represent \mathcal{A}_L in the form

$$\mathcal{A}_L = \{T^{(1)}, T^{(2)}, \dots, T^{(m_L)}\}.$$

Then, by (6.9) and (6.10), for every $i = 1, \dots, m_L$,

$$(6.24) \quad \text{diam}\{A_L, B_L, A_{T^{(i)}}, B_{T^{(i)}}\} \sim \text{diam } Q_L$$

and

$$(6.25) \quad \{A_L, B_L, A_{T^{(i)}}, B_{T^{(i)}}\} \subset \gamma Q_L.$$

By subdividing each edge of the cube Q_L into m_L equal parts we can partition this cube into a family $\{\widetilde{K}_1, \widetilde{K}_2, \dots, \widetilde{K}_{M_L}\}$ consisting of $M_L := m_L^n$ congruent cubes of diameter $\text{diam } Q_L / m_L$. We put

$$(6.26) \quad K_i = \frac{1}{8} \widetilde{K}_i, \quad i = 1, \dots, M_L.$$

Since

$$\text{diam } K_i = \text{diam } Q_L / (8m_L), \quad i = 1, \dots, m_L,$$

for some absolute constant $\gamma = \gamma(n)$ we have

$$Q_L \subset \gamma K_i, \quad i = 1, \dots, m_L.$$

Hence, by (6.24) and (6.25), for every $i = 1, \dots, m_L$,

$$(6.27) \quad \text{diam}\{A_L, B_L, A_{T^{(i)}}, B_{T^{(i)}}\} \sim \text{diam } K_i$$

and

$$(6.28) \quad \{A_L, B_L, A_{T^{(i)}}, B_{T^{(i)}}\} \subset \gamma K_i.$$

Finally we put

$$K(T_L, T^{(i)}) := K_i, \quad i = 1, \dots, m_L.$$

Properties (6.27) and (6.28) show that this formula defines the required mapping $(T, T') \rightarrow K(T, T')$ for all pairs (T_L, T) where $T \in \mathcal{A}_L$.

In the same way we define the mapping $(T, T') \rightarrow K(T, T')$ for all pairs (T_L, T) with $T \in \mathcal{A}^{(L)}$.

Thus we have defined the required mapping on the family

$$\{(T_L, T) : T \in \mathcal{BR}_E, T \longleftrightarrow T_L\}$$

of bridges connected to the (interior) bridge T_L .

Let us apply this procedure to every lacuna $L \in \mathcal{L}_E$. We obtain a mapping $(T, T') \rightarrow K(T, T')$ which to every element of the set

$$\{(T, T') \in \mathcal{BR}_E \times \mathcal{BR}_E : T \longleftrightarrow T', T \neq T'\}$$

assigns a cube

$$K(T, T') \in \mathcal{K}_E.$$

Here, by (6.26), \mathcal{K}_E is a family of cubes defined as follows:

$$\mathcal{K}_E := \frac{1}{8} \tilde{\mathcal{K}}_E = \{\frac{1}{8} \tilde{K} : \tilde{K} \in \tilde{\mathcal{K}}_E\}$$

where

$$\tilde{\mathcal{K}}_E = \bigcup_{L \in \mathcal{L}_E} \{\tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_{M_L}\}.$$

By (6.27) and (6.28), this mapping satisfies the required conditions (6.21) and (6.22). Furthermore, since the Whitney cubes are non-overlapping, the cubes of the family

$$\tilde{\mathcal{K}}_E = \bigcup_{L \in \mathcal{L}_E} \{\tilde{K}_1, \tilde{K}_2, \dots, \tilde{K}_{M_L}\}$$

are non-overlapping as well, so that the cubes of the family $\mathcal{K}_E := \frac{1}{8} \tilde{\mathcal{K}}_E$ are *well-separated*. (See Definition 6.5.)

However, in general we can not guarantee that this mapping is well-defined. In fact, if $T, T' \in \mathcal{BR}_E, T \longleftrightarrow T'$, is a pair of *connected interior bridges*, the cubes $K(T, T')$ and $K(T', T)$ may be different. To avoid this situation, in this case to the pair (T, T') we *simply assign one of these cubes* (no matter which one). As a result we obtain a *well-defined* one-to-one mapping satisfying all the conditions of the proposition.

The proposition is proved. ■

7. A lacunary Whitney-type extension operator.

We return to the proof of the sufficiency part of Theorem 1.2. Thus in this and the next section E is a finite subset of \mathbf{R}^2 and $p \in (2, \infty)$.

Our aim is to prove a stronger result, Theorem 7.1, which immediately implies the sufficiency in Theorem 1.2.

Theorem 7.1 *Let E be a finite subset of \mathbf{R}^2 and let f be a function on E . Then*

$$(7.1) \quad \|f\|_{L_p^2(\mathbf{R}^2)|_E} \leq C(p) \lambda^{\frac{1}{p}}$$

provided λ is a positive constant which satisfies all of the following conditions for a certain absolute positive constant γ :

(a). The condition (i) of Theorem 1.2 holds;

(b). Let \mathcal{Q} and \mathcal{K} be finite families of pairwise disjoint squares. Assume that to each square $K \in \mathcal{K}$ we have arbitrarily assigned a triangle $\Delta(K)$ in E such that

$$\Delta(K) \subset \gamma K \quad \text{and} \quad \text{diam } K \leq \gamma \text{diam } \Delta(K).$$

Suppose that to each square $Q \in \mathcal{Q}$ we have arbitrarily assigned a pair of squares $Q', Q'' \in \mathcal{Q}$ such that $Q' \cup Q'' \subset \gamma Q$ and

$$(7.2) \quad (\text{diam } Q')^{p-2} \sigma_p(Q'; \mathcal{K}) + (\text{diam } Q'')^{p-2} \sigma_p(Q''; \mathcal{K}) \leq 1.$$

Then the following inequality

$$(7.3) \quad \sum_{Q \in \mathcal{Q}} \left(\frac{\text{diam } Q' \text{diam } Q''}{\text{diam } Q} \right)^{p-2} S_p(f : Q', Q''; \mathcal{K}) \leq \lambda$$

holds.

We recall that the quantities σ_p and S_p have been defined in Section 1, see formulation of Theorem 1.2.

Proof. We prove Theorem 7.1 in several stages. At the first stage given a family of affine interpolating polynomials we construct a Whitney-type extension of the function f from E to all of \mathbf{R}^2 . Then we estimate the L_p^2 -norm of this extension via oscillations of these polynomials on contacting Whitney squares.

Let

$$\{P_L \in \mathcal{P}_1 : L \in \mathcal{L}_E\}$$

be a family of affine polynomials such that for every lacuna $L \in \mathcal{L}_E$

$$(7.4) \quad P_L(A_L) = f(A_L), \quad P_L(B_L) = f(B_L).$$

Let $L \in \mathcal{L}_E$ be a lacuna of Whitney squares. To every square $Q \in L$ we assign an affine polynomial

$$P^{(Q)} := P_L.$$

Then we construct an extension of the function f using the classical Whitney formula

$$(7.5) \quad F(x) := \begin{cases} f(x), & x \in E, \\ \sum_{Q \in W_E} \varphi_Q(x) P^{(Q)}(x), & x \in \mathbf{R}^2 \setminus E. \end{cases}$$

Here as usual $\{\varphi_Q : Q \in W_E\}$ denotes a smooth partition of unity subordinated to the Whitney decomposition W_E . Let us recall its main properties, see, e.g. [43], Ch. 6.

Lemma 7.2 *The family of functions $\{\varphi_Q : Q \in W_E\}$ has the following properties:*

- (a). $\varphi_Q \in C^\infty(\mathbf{R}^2)$ and $0 \leq \varphi_Q \leq 1$ for every $Q \in W_E$;
- (b). $\text{supp } \varphi_Q \subset Q^* (:= \frac{9}{8}Q)$, $Q \in W_E$;
- (c). $\sum \{\varphi_Q(x) : Q \in W_E\} = 1$ for every $x \in \mathbf{R}^2 \setminus E$;
- (d). $|D^\beta \varphi_Q(x)| \leq C(\text{diam } Q)^{-|\beta|}$ for every $Q \in W_E$, every $x \in \mathbf{R}^2$ and every multiindex β of order $|\beta| \leq 2$.

Given lacunae $L, L' \in \mathcal{L}_E$ we introduce a square

$$(7.6) \quad Q(L, L') := Q(A_L, d(L, L')).$$

Recall that

$$d(L, L') := \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\}.$$

Proposition 7.3 *The L_p^2 -norm of the extension F satisfies the following inequality:*

$$\|F\|_{L_p^2(\mathbf{R}^2)}^p \leq C \sum \left\{ d(L, L')^{2-2p} \max_{Q(L, L')} |P_L - P_{L'}|^p : L, L' \in \mathcal{L}_E, L \leftrightarrow L' \right\}.$$

Proof. Since every point $z \in E$ is isolated, there exists a (unique) true lacuna $L \in \mathcal{L}_E$ such that $V_L = \{z\}$. Hence $A_L = z$ so that, by (7.4), $P_L(z) = f(z)$. Since in this case the set $\{\cup Q : Q \in L\}$ contains a certain neighborhood of z , by the formula (7.5), F coincides with P_L on this neighborhood.

On the other hand, by the same formula, $F \in C^\infty(\mathbf{R}^2 \setminus E)$ so that $F \in C^\infty(\mathbf{R}^2)$. This shows that distributional partial derivatives of F can be identified with its regular derivatives. Let us estimate such a derivative of F of order two on a Whitney square.

Let $K \in W_E$ be a Whitney square which belongs to a lacuna $L \in \mathcal{L}_E$. Let $x \in K$ and let α be a multiindex, $|\alpha| = 2$. Then, by the formula (7.5), Lemma 7.2 and Lemma 4.2,

$$\begin{aligned} |D^\alpha F(x)| &= |D^\alpha(F(x) - P_L(x))| = \left| D^\alpha \left(\sum \{ \varphi_Q(x) (P^{(Q)}(x) - P_L(x)) : Q \in W_E \} \right) \right| \\ &= \left| D^\alpha \left(\sum \{ \varphi_Q(x) (P^{(Q)}(x) - P_L(x)) : Q \in W_E, Q^* \ni x \} \right) \right| \\ &\leq \sum \{ |D^\alpha(\varphi_Q(P^{(Q)} - P_L))(x)| : Q \in W_E, Q^* \ni x \}. \end{aligned}$$

Given a square $Q \in W_E$ such that $Q^* \ni x$ let us estimate the quantity

$$I_Q(x) := |D^\alpha(\varphi_Q(P^{(Q)} - P_L))(x)|.$$

We have

$$I_Q(x) \leq C \sum_{|\beta| \leq 2} |D^{\alpha-\beta} \varphi_Q(x)| \cdot |D^\beta(P^{(Q)} - P_L)(x)|$$

so that, by Lemma 7.2,

$$I_Q(x) \leq C \sum_{|\beta| \leq 2} (\text{diam } Q)^{|\beta| - |\alpha|} \max_{Q^*} |D^\beta(P^{(Q)} - P_L)|.$$

By Markov's inequality,

$$\max_{Q^*} |D^\beta(P^{(Q)} - P_L)| \leq C(\text{diam } Q^*)^{-|\beta|} \max_{Q^*} |P^{(Q)} - P_L|.$$

Clearly, for every polynomial $P \in \mathcal{P}_1$, every square $Q \subset \mathbf{R}^2$ and every $\gamma > 1$ we have

$$(7.7) \quad \max_{\gamma Q} |P| \leq C(\gamma) \max_Q |P|.$$

Hence

$$\max_{Q^*} |D^\beta(P^{(Q)} - P_L)| \leq C(\text{diam } Q)^{-|\beta|} \max_Q |P^{(Q)} - P_L|$$

so that

$$I_Q(x) \leq C(\text{diam } Q)^{-|\alpha|} \max_Q |P^{(Q)} - P_L| = C(\text{diam } Q)^{-2} \max_Q |P^{(Q)} - P_L|.$$

We obtain

$$\begin{aligned} |D^\alpha F(x)| &\leq C \sum \left\{ (\text{diam } Q)^{-2} \max_Q |P^{(Q)} - P_L| : Q \in W_E, Q^* \ni x \right\} \\ &\leq C \sum \left\{ (\text{diam } Q)^{-2} \max_Q |P^{(Q)} - P_L| : Q \in W_E, Q^* \cap K \neq \emptyset \right\} \end{aligned}$$

so that, by part (3) of Lemma 4.2,

$$|D^\alpha F(x)| \leq C \sum \left\{ (\text{diam } Q)^{-2} \max_Q |P^{(Q)} - P_L| : Q \in W_E, Q \cap K \neq \emptyset \right\}.$$

We let J_K denote a family of Whitney squares

$$J_K := \{Q \in W_E : Q \cap K \neq \emptyset\}.$$

We note that, by part (1) of Lemma 4.2, $\text{diam } Q \sim \text{diam } K$ for every square $Q \in J_K$. Hence

$$|D^\alpha F(x)| \leq C(\text{diam } K)^{-2} \sum_{Q \in J_K} \max_Q |P^{(Q)} - P_L|.$$

By part (2) of Lemma 4.2, $\#J_K \leq C$ so that

$$|D^\alpha F(x)| \leq C(\text{diam } K)^{-2} \max_{Q \in J_K} \max_Q |P^{(Q)} - P_L|.$$

We let L^{cont} denote a family of *contacting* squares of the lacuna L :

$$L^{cont} := \{Q \in L : \exists L' \in \mathcal{L}_E, L' \neq L, Q' \in L' \text{ such that } Q' \cap Q \neq \emptyset\}.$$

Note that if the square K is not a contacting square, i.e., $K \in L \setminus L^{(cont)}$, then $J_K \subset L$ so that $P^{(Q)} = P_L$ for every $Q \in J_K$. Hence

$$D^\alpha F(x) = 0 \quad \text{for every } K \in L \setminus L^{(cont)} \text{ and every } x \in K.$$

Let $K \in L^{(cont)}$ be a contacting square. Recall that $P^{(Q)} = P_L$ for every square $Q \in L$. Therefore there exists a contacting lacuna $L_K \in \mathcal{L}_E$, $L_K \leftrightarrow L$, $L_K \neq L$, and a square $\tilde{K} \in L_K$ such that $\tilde{K} \in J_K$ and

$$\max_{Q \in J_K} \max_Q |P^{(Q)} - P_L| = \max_{\tilde{K}} |P^{(\tilde{K})} - P_L| = \max_{\tilde{K}} |P_{L_K} - P_L|.$$

(In particular \tilde{K} is a contacting square for K .) Hence

$$|D^\alpha F(x)| \leq C (\text{diam } K)^{-2} \max_{\tilde{K}} |P_{L_K} - P_L|, \quad x \in K.$$

Integrating this inequality to the power p on the square K we obtain

$$\int_K |D^\alpha F(x)|^p dx \leq C (\text{diam } K)^{-2p+2} \max_{\tilde{K}} |P_{L_K} - P_L|^p.$$

Note that, by Proposition 5.9, $\text{diam } K \sim d(L, L_K)$. Furthermore,

$$\{A_L, B_L, A_{L_K}, B_{L_K}\} \subset \gamma K$$

so that the square $Q(L, L_K) = Q(A_L, d(L, L_K))$ have common points with K . Hence $K \subset \gamma_1 Q(L, L_K)$ with some absolute constant γ_1 .

We obtain

$$\int_K |D^\alpha F(x)|^p dx \leq C d(L, L_K)^{-2p+2} \max_{\gamma_1 Q(L, L_K)} |P_{L_K} - P_L|^p.$$

By (7.7),

$$\max_{\gamma_1 Q(L, L_K)} |P_{L_K} - P_L| \leq C \max_{Q(L, L_K)} |P_{L_K} - P_L|$$

so that

$$\int_K |D^\alpha F(x)|^p dx \leq C d(L, L_K)^{-2p+2} \max_{Q(L, L_K)} |P_{L_K} - P_L|^p.$$

Hence

$$\sum_{K \in L} \int_K |D^\alpha F(x)|^p dx \leq C \sum \left\{ d(L, L')^{-2p+2} \max_{Q(L, L')} |P_{L'} - P_L|^p : L' \in \mathcal{L}_E, L' \leftrightarrow L \right\}$$

which implies that

$$\begin{aligned} \|F\|_{L_p^2(\mathbf{R}^2)}^p &= \int_{\mathbf{R}^2} |D^\alpha F(x)|^p dx \leq \sum_{L \in \mathcal{L}_E} \sum_{K \in L} \int_K |D^\alpha F(x)|^p dx \\ &\leq C \sum_{L \in \mathcal{L}_E} \sum_{L' \leftrightarrow L} d(L, L')^{-2p+2} \max_{Q(L, L')} |P_{L'} - P_L|^p \end{aligned}$$

proving the proposition. ■

We follow to the scheme of the proof described in Section 2. Our next goal is to express the Sobolev norm of the extension F defined by the formula (7.5) via *the gradients* of interpolating polynomials.

Let $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ be a mapping which to every bridge $T \in \mathcal{BR}_E$ (interior or exterior) assigns a vector in \mathbf{R}^2 . Suppose that for every bridge $T \in \mathcal{BR}_E$ with the ends at points $A^{[T]}, B^{[T]} \in E$ the following equality

$$(7.8) \quad \langle g(T), A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]})$$

holds. (Recall that $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbf{R}^2 .)

Let $L \in \mathcal{L}_E$ be a lacuna and let $P_L \in \mathcal{P}_1$ be an affine polynomial defined by the following formula

$$(7.9) \quad P_L(x) := f(A_L) + \langle g(T_L), x - A_L \rangle, \quad x \in \mathbf{R}^2.$$

Recall that T_L denotes the interior bridge $T_L = (A_L, B_L) \in \mathcal{BR}_E$. Clearly, by (7.8),

$$(7.10) \quad P_L(A_L) = f(A_L), \quad P_L(B_L) = f(B_L).$$

Let us also note that $g(T_L) = \nabla P_L$.

Let $T, T' \in \mathcal{BR}_E$ be a pair of bridges with the ends at points $\{A^{[T]}, B^{[T]}\}$ and $\{A^{[T']}, B^{[T']}\}$ correspondingly. Let

$$(7.11) \quad D(T, T') := \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}.$$

Proposition 7.4 *Let F be the extension of the function f defined by the formula (7.5) where for every lacuna $L \in \mathcal{L}_E$ the polynomial $P_L \in \mathcal{P}_1$ is determined by (7.9). Then*

$$\|F\|_{L_p^2(\mathbf{R}^2)}^p \leq C \sum \{D(T, T')^{2-p} \|g(T) - g(T')\|^p : T, T' \in \mathcal{BR}_E, T \leftrightarrow T'\}.$$

Proof. Let $L, L' \in \mathcal{L}_E$, $L \leftrightarrow L'$, be contacting lacunae. Let

$$T_L = (A_L, B_L) \quad \text{and} \quad T_{L'} = (A_{L'}, B_{L'})$$

be interior bridges of L and L' with the ends $\{A_L, B_L\}$ and $\{A_{L'}, B_{L'}\}$.

Suppose that these bridges are *connected to each other by an exterior bridge*

$$T = \{C(L, L'), C(L', L)\}.$$

Thus $T_L \rightsquigarrow T$ and $T_{L'} \rightsquigarrow T$ and

$$C(L, L') \in \{A_L, B_L\} \quad \text{and} \quad C(L', L) \in \{A_{L'}, B_{L'}\}.$$

Let us estimate the quantity

$$I(L, L') := d(L, L')^{2-2p} \max_{Q(L, L')} |P_L - P_{L'}|^p$$

which appears in the righthand side of the inequality of Proposition 7.3. Let

$$(7.12) \quad P^{[T]}(x) := f(C(L, L')) + \langle g(T), x - C(L, L') \rangle, \quad x \in \mathbf{R}^2.$$

Note that, by (7.8),

$$\langle g(T), C(L, L') - C(L', L) \rangle = f(C(L, L')) - f(C(L', L))$$

so that

$$P^{[T]}(C(L, L')) = f(C(L, L')) \quad \text{and} \quad P^{[T]}(C(L', L)) = f(C(L', L)).$$

Since $C(L, L') \in \{A_L, B_L\}$, by (7.10), $P_L(C(L, L')) = f(C(L, L'))$. Since $P_L \in \mathcal{P}_1$, it can be represented in the form

$$P_L(x) := f(C(L, L')) + \langle \nabla P_L, x - C(L, L') \rangle, \quad x \in \mathbf{R}^2.$$

Hence

$$P_L(x) := f(C(L, L')) + \langle g(T_L), x - C(L, L') \rangle$$

so that, by (7.12), for every $x \in \mathbf{R}^2$ we have

$$|P_L(x) - P^{[T]}(x)| = |\langle g(T_L) - g(T), x - C(L, L') \rangle|.$$

This implies the following inequality:

$$(7.13) \quad |P_L(x) - P^{[T]}(x)| \leq C \|g(T_L) - g(T)\| \|x - C(L, L')\|, \quad x \in \mathbf{R}^2.$$

Recall that

$$(7.14) \quad D(T_L, T_{L'}) = \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\},$$

see (7.11), and

$$Q(L, L') = Q(A_L, d(L, L'))$$

where $d(L, L') = \text{diam}\{A_L, B_L, A_{L'}, B_{L'}\}$, see (7.6). (Thus $d(L, L') = D(T_L, T_{L'})$.)

In particular,

$$A_L, B_L, A_{L'}, B_{L'} \in Q(L, L')$$

so that $C(L, L'), C(L', L) \in Q(L, L')$ as well. Hence, by (7.13),

$$\max_{Q(L, L')} |P_L - P^{[T]}| \leq C \text{diam } Q(L, L') \|g(T_L) - g(T)\| \leq C d(L, L') \|g(T_L) - g(T)\|.$$

In the same way we obtain the following inequality

$$\max_{Q(L, L')} |P_{L'} - P^{[T]}| \leq C d(L, L') \|g(T_{L'}) - g(T)\|.$$

Hence

$$\begin{aligned} I(L, L') &= d(L, L')^{2-2p} \max_{Q(L, L')} |P_L - P_{L'}|^p \\ &\leq C d(L, L')^{2-p} \{\|g(T_L) - g(T)\|^p + \|g(T_{L'}) - g(T)\|^p\}. \end{aligned}$$

But, by (7.14),

$$D(T_L, T) = \text{diam}\{A_L, B_L, C(L', L)\} \leq D(T_L, T_{L'}) = d(L, L')$$

and

$$D(T_{L'}, T) = \text{diam}\{A_{L'}, B_{L'}, C(L', L)\} \leq D(T_L, T_{L'}) = d(L, L')$$

so that

$$I(L, L') \leq C \{D(T_L, T)^{2-p} \|g(T_L) - g(T)\|^p + D(T_{L'}, T)^{2-p} \|g(T_{L'}) - g(T)\|^p\}.$$

We obtain the same estimate of $I(L, L')$ whenever *the interior bridges T_L and $T_{L'}$ are connected*. In this case we can simply put $T := T_{L'}$ and repeat the same proof as for the previous case where T_L and $T_{L'}$ are connected by an exterior bridge.

It remains to replace in the inequality of Proposition 7.3 the quantity $I(L, L')$ by its estimate via the mapping g given above, and the proposition follows. \blacksquare

We turn to the next step of the proof of Theorem 7.1. We shall estimate the L_p^2 -norm of the extension F of the function f via L_p -norm of an additional parameter, a function $h : \mathbf{R}^2 \rightarrow \mathbf{R}_+$. We will see that averages of this function on certain squares majorize distances between values of the mapping $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ on connected bridges, see Proposition 7.6 below.

The proof of this proposition uses on an auxiliary result related to the theory of Muckenhoupt's weights. We recall, see, e.g. [21], that a non-negative function $w \in L_{1,loc}(\mathbf{R}^2)$ is said to be A_1 -weight if there exists $\lambda > 0$ such that for every square $Q \in \mathbf{R}^2$ the following inequality

$$\frac{1}{|Q|} \int_Q w(u) du \leq \lambda \text{ess inf } w$$

holds. We put $\|w\|_{A_1} = \inf \lambda$.

Clearly, a weight $w \in A_1$ if and only if

$$\mathcal{M}[w] \leq \lambda w(x) \quad \text{a.e. on } \mathbf{R}^2,$$

and

$$\|w\|_{A_1} \sim \text{ess sup}_{\mathbf{R}^2} \frac{\mathcal{M}[w](x)}{w(x)}.$$

Recall that

$$\mathcal{M}[h](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |h| dx, \quad x \in \mathbf{R}^2,$$

denotes the Hardy-Littlewood maximal function of a function $h \in L_{1,loc}(\mathbf{R}^2)$. We also note the following property of a weight $w \in A_1$: if K, Q are two squares in \mathbf{R}^2 such that $K \subset Q$, then

$$(7.15) \quad \frac{1}{|Q|} \int_Q w(u) du \leq \|w\|_{A_1} \frac{1}{|K|} \int_K w(u) du.$$

(In other words, the average of a function $w \in A_1$ is a quasi-monotone non-increasing function of a square.)

The following remarkable result of Coifman and Rochberg [8] presents an important property of A_1 -weights.

Theorem 7.5 *Let $w \in L_{1,loc}(\mathbf{R}^2)$, and let $\mathcal{M}[w](x) < \infty$ a.e. Then $\mathcal{M}[w]^\theta \in A_1(\mathbf{R}^2)$ for every $0 < \theta < 1$. Furthermore, $\|\mathcal{M}[w]^\theta\|_{A_1} \leq C(\theta)$.*

Let $q = (p+2)/2$; thus $2 < q < p$. Let $T, T' \in \mathcal{BR}_E$ be a pair of bridges with the ends at points $\{A^{[T]}, B^{[T]}\}$ and $\{A^{[T']}, B^{[T']}\}$ respectively. By $Q(T, T')$ we denote a square

$$(7.16) \quad Q(T, T') = Q(A^{[T]}, D(T, T')).$$

Recall that $D(T, T') := \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}$, see (7.11).

Proposition 7.6 *Let $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ be a mapping satisfying condition (7.8), and let F be the extension of f defined by (7.5) where the polynomials $P_L, L \in \mathcal{L}_E$, are given by the formula (7.9).*

Let $\gamma > 1$ and let $h : \mathbf{R}^2 \rightarrow \mathbf{R}_+$ be a non-negative function such that $h \in L_p(\mathbf{R}^2)$. Suppose that for every pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \rightsquigarrow T'$, the following inequality

$$(7.17) \quad \|g(T) - g(T')\| \leq \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h^q(z) dz \right)^{\frac{1}{q}}$$

holds. Then $F \in L_p^2(\mathbf{R}^2)$ and

$$\|F\|_{L_p^2(\mathbf{R}^2)} \leq C(p, \gamma) \|h\|_{L_p(\mathbf{R}^2)}.$$

Proof. Let $s := (q+p)/2$; thus $q < s < p$. We let $\tilde{h} : \mathbf{R}^2 \rightarrow \mathbf{R}_+$ denote a function

$$\tilde{h}(z) := (\mathcal{M}[h^s](z))^{\frac{1}{s}}, \quad z \in \mathbf{R}^2.$$

By the Lebesgue theorem, $h^s \leq \mathcal{M}[h^s]$ a.e. so that

$$(7.18) \quad h \leq \tilde{h} \quad \text{a.e. on } \mathbf{R}^2.$$

Since $p/s > 1$, by the Hardy-Littlewood maximal theorem,

$$\|\tilde{h}\|_{L_p(\mathbf{R}^2)} = \left(\int_{\mathbf{R}^2} (\mathcal{M}[h^s])^{\frac{p}{s}} dz \right)^{\frac{1}{p}} \leq C(p) \left(\int_{\mathbf{R}^2} (h^s)^{\frac{p}{s}} dz \right)^{\frac{1}{p}}$$

proving that

$$(7.19) \quad \|\tilde{h}\|_{L_p(\mathbf{R}^2)} \leq C\|h\|_{L_p(\mathbf{R}^2)}.$$

Since $\theta := q/s < 1$, by Theorem 7.5, the function

$$\tilde{h}^q = (\mathcal{M}[h^s])^{\frac{q}{s}} = (\mathcal{M}[h^s])^\theta$$

belongs to the class $A_1(\mathbf{R}^2)$ and $\|\tilde{h}^s\|_{A_1(\mathbf{R}^2)} \leq C(\theta) = C(p)$. Hence, by (7.15),

$$(7.20) \quad \frac{1}{|Q|} \int_Q \tilde{h}^s dz \leq C \frac{1}{|K|} \int_K \tilde{h}^s dz$$

provided K, Q are arbitrary squares in \mathbf{R}^2 such that $K \subset Q$.

By (7.18) and (7.17), for every pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \rightsquigarrow T'$, the following inequality

$$(7.21) \quad \|g(T) - g(T')\| \leq \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} \tilde{h}^q dz \right)^{\frac{1}{q}}$$

holds. Let

$$\mathcal{BR}_E \ni T, T' \mapsto K(T, T') \in \mathcal{K}_E$$

be a one-to-one mapping constructed in Proposition 6.6. To every pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \rightsquigarrow T'$, this mapping assigns a square $K(T, T')$ which belongs to a family \mathcal{K}_E of pairwise disjoint squares satisfying conditions (6.21) and (6.22).

Let us compare the square $\gamma Q(T, T') = Q(A^{[T]}, \gamma D(T, T'))$ with the square $K(T, T')$. By (6.21),

$$(7.22) \quad \text{diam } K(T, T') \sim \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} = D(T, T') = \frac{1}{2} \text{diam } Q(T, T').$$

In particular, $|Q(T, T')| \sim |K(T, T')|$.

Note that, by (6.22), the point $A^{[T]} \in \gamma_1 K(T, T')$ where γ_1 is an absolute constant. Hence

$$\gamma Q(T, T') \cap (\gamma_1 K(T, T')) \neq \emptyset.$$

Combining this property with equivalence (7.22) we conclude that for some absolute constant $\gamma_2 = \gamma_2(\gamma) \geq \gamma_1$ the following inclusion

$$\gamma Q(T, T') \subset \tilde{K} := \gamma_2 K(T, T')$$

holds. Hence

$$\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} \tilde{h}^q dz \leq C \frac{1}{|\tilde{K}|} \int_{\tilde{K}} \tilde{h}^q dz.$$

Since $K(T, T') \subset \tilde{K}$, by (7.20),

$$\frac{1}{|\tilde{K}|} \int_{\tilde{K}} \tilde{h}^q dz \leq C(p) \frac{1}{|K(T, T')|} \int_{K(T, T')} \tilde{h}^q dz.$$

Combining this inequality with (7.21) and (7.22) we obtain

$$\|g(T) - g(T')\| \leq C \operatorname{diam} K(T, T') \left(\frac{1}{|K(T, T')|} \int_{K(T, T')} \tilde{h}^q dz \right)^{\frac{1}{q}}$$

provided $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$. Since $2 < q < p$, by the Hölder inequality,

$$\begin{aligned} \|g(T) - g(T')\| &\leq C \operatorname{diam} K(T, T') \left(\frac{1}{|K(T, T')|} \int_{K(T, T')} \tilde{h}^p dz \right)^{\frac{1}{p}} \\ &= C (\operatorname{diam} K(T, T'))^{1-2/p} \left(\int_{K(T, T')} \tilde{h}^p dz \right)^{\frac{1}{p}}. \end{aligned}$$

Since $D(T, T') \sim \operatorname{diam} K(T, T')$, see (7.22), we obtain

$$\|g(T) - g(T')\|^p \leq C D(T, T')^{p-2} \int_{K(T, T')} \tilde{h}^p dz.$$

Finally, by Proposition 7.4,

$$\begin{aligned} \|F\|_{L_p^2(\mathbf{R}^2)}^p &\leq C \sum \left\{ D(T, T')^{2-p} \|g(T) - g(T')\|^p : T, T' \in \mathcal{BR}_E, T \rightsquigarrow T' \right\} \\ &\leq C \sum \left\{ \int_{K(T, T')} \tilde{h}^p dz : T, T' \in \mathcal{BR}_E, T \rightsquigarrow T' \right\} \\ &= C \sum_{K \in \mathcal{K}_E} \int_K \tilde{h}^p dz. \end{aligned}$$

Since the family \mathcal{K}_E consists of pairwise disjoint squares, we obtain

$$\|F\|_{L_p^2(\mathbf{R}^2)}^p \leq C \int_{\mathbf{R}^2} \tilde{h}^p dz = C \|\tilde{h}\|_{L_p(\mathbf{R}^2)}^p$$

so that, by (7.19), $\|F\|_{L_p^2(\mathbf{R}^2)}^p \leq C \|h\|_{L_p(\mathbf{R}^2)}^p$.

The proposition is proved. ■

8. Sobolev-type selections of set-valued mappings and a decomposition of the sum $\vec{\Sigma} := \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$.

8.1. Sobolev-type selections. The extension criterion formulated in Proposition 7.6 admits a geometrical reformulation in terms of *set-valued mappings and their selections*.

Let $f : E \rightarrow \mathbf{R}$ be a function on the set E and let $T \in \mathcal{BR}_E$ be a bridge (interior or exterior) with the ends at points $A^{[T]}, B^{[T]} \in E$. We let $G_f(T)$ denote a straight line in \mathbf{R}^2

$$(8.1) \quad G_f(T) := \{z \in \mathbf{R}^2 : \langle z, A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]})\}.$$

Let $\text{Aff}(\mathbf{R}^2)$ be the family of all straight lines in \mathbf{R}^2 . The formula (8.1) defines a mapping

$$G_f : \mathcal{BR}_E \rightarrow \text{Aff}(\mathbf{R}^2)$$

which to every bridge $T \in \mathcal{BR}_E$ assigns a *subset* of \mathbf{R}^2 , the straight line $G_f(T)$. We refer to G_f as a *set-valued mapping*.

Let $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ be a (regular) mapping such that for every bridge $T \in \mathcal{BR}_E$ with the ends at points $A^{[T]}, B^{[T]} \in E$ the following equality

$$\langle g(T), A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]})$$

holds. By (8.1), this property can be reformulated in the following way:

$$g(T) \in G_f(T) \quad \text{for every bridge } T \in \mathcal{BR}_E.$$

We refer to the mapping g as a *selection* of the set-valued mapping G_f .

We are interested in a special type of selections which satisfy the inequality (7.17) from Proposition 7.6:

$$(8.2) \quad \|g(T) - g(T')\| \leq \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h^q(z) dz \right)^{\frac{1}{q}}$$

provided $\gamma \geq 1$ is an absolute constant, $2 < q < p$, h is a non-negative $L_p(\mathbf{R}^2)$ -function, and T and T' are arbitrary connected bridges from \mathcal{BR}_E . This inequality is a natural analog of the Sobolev-Poincaré inequality (3.2). This analogy motivates us to refer to a selection g satisfying (7.17) as a *Sobolev-type selection* (with respect to the function h) of the set-valued mapping G_f .

Using this terminology we can formulate the following criterion for the trace space $L_p^2(\mathbf{R}^2)|_E$.

Claim 8.1 *Let $2 < q < p < \infty$ and let f be a function defined on a finite subset $E \subset \mathbf{R}^2$. Then the following equivalence*

$$\|f\|_{L_p^2(\mathbf{R}^2)|_E} \sim \inf \|h\|_{L_p(\mathbf{R}^2)}$$

holds with constants depending only on p and q . Here the infimum is taken over all non-negative functions $h \in L_p(\mathbf{R}^2)$ such that the set-valued mapping $G_f : \mathcal{BR}_E \rightarrow \text{Aff}(\mathbf{R}^2)$ has a Sobolev-type selection with respect to h .

Proof. The inequality

$$\|f\|_{L_p^2(\mathbf{R}^2)|_E} \leq C \inf \|h\|_{L_p(\mathbf{R}^2)}$$

directly follows from Proposition 7.6.

Let us prove the converse inequality. Let $\gamma \geq 1$ be an arbitrary constant. Let $F \in L_p^2(\mathbf{R}^2)$, $F|_E = f$, be an arbitrary extension of f and let $T \in \mathcal{BR}_E$ be a bridge with the ends at the points $A^{[T]}, B^{[T]} \in E$. By the Lagrange theorem there exists a point $Z_T \in [A^{[T]}, B^{[T]}]$ such that

$$(8.3) \quad \langle \nabla F(Z_T), A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]}).$$

We put

$$g(T) := \nabla F(Z_T), \quad T \in \mathcal{BR}_E.$$

Then, by (8.3), g is a *selection* of the set-valued mapping G_f , see (8.1).

Now let us consider an arbitrary pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \rightsquigarrow T'$. Clearly the square

$$Q(T, T') = Q(A^{[T]}, D(T, T'))$$

where $D(T, T') = \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}$, contains the set $\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}$. Hence $Z_T, Z_{T'} \in Q(T, T')$ as well so that, by the Sobolev-Poincaré inequality (3.2),

$$\begin{aligned} \|g(T) - g(T')\| &= \|\nabla F(Z_T) - \nabla F(Z_{T'})\| \\ &\leq C \text{diam } Q(T, T') \left(\frac{1}{|Q(T, T')|} \int_{Q(T, T')} (\nabla^2 F)^q dz \right)^{\frac{1}{q}} \\ &\leq C(\gamma) \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} (\nabla^2 F)^q dz \right)^{\frac{1}{q}}. \end{aligned}$$

This shows that g satisfies inequality (8.2) with $h := C(\gamma) \nabla^2 F$. Hence

$$\|h\|_{L_p(\mathbf{R}^2)} = C \|\nabla^2 F\|_{L_p(\mathbf{R}^2)} = C \|F\|_{L_p^2(\mathbf{R}^2)}.$$

Taking the infimum in this inequality over all functions $F \in L_p^2(\mathbf{R}^2)$ such that $F|_E = f$ we obtain the required inequality

$$\inf \|h\|_{L_p(\mathbf{R}^2)} \leq C \|f\|_{L_p^2(\mathbf{R}^2)|_E}$$

with $C = C(q, \gamma)$. The claim is proved. ■

Remark 8.2 Given $q \in (2, p)$ consider a function

$$\delta_h(T, T') := \text{diam } Q(T, T') \left(\frac{1}{|Q(T, T')|} \int_{Q(T, T')} h^q(z) dz \right)^{\frac{1}{q}}, \quad T, T' \in \mathcal{BR}_E.$$

Functions of such a kind have been studied in the author's paper [41]. Using the methods of this work one can show that the function δ is equivalent to a certain *metric* $\tilde{\delta}_h$ on \mathcal{BR}_E provided $h^q \in A_1(\mathbf{R}^2)$.

Thus in this case inequality (8.2) is equivalent to the inequality

$$\|g(T) - g(T')\| \leq \tilde{\delta}_h(T, T'), \quad T, T' \in \mathcal{BR}_E.$$

In other words, the mapping $g : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ is *Lipschitz* with respect to the metric $\tilde{\delta}$ so that g is a *Lipschitz selection* of the set valued mapping G_f . Our aim is to *find the order of magnitude of the minimal $L_p(\mathbf{R}^2)$ -norm of the function h provided there exists a Lipschitz selection of G_f with respect to the metric $\tilde{\delta}_h$.*

This type of selection problems is a generalization of the so-called Lipschitz selection problem. We refer the reader to the papers [36, 35, 37] and references therein for numerous results related to this problem and techniques for obtaining them. In particular, our approach to the Sobolev-type selection problem which we develop in the next subsection is a generalization of ideas and methods suggested in these papers for solution to the Lipschitz selection problem. \triangleleft

8.2 The mapping $\mathcal{T}(f)$ and its norm in $\vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$. Following the scheme of the proof given in Section 2, in this subsection we show that the mapping $\mathcal{T}(f)$ defined in Section 2 belongs to the space $\vec{\Sigma} := \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$ and its norm in this space is bounded by $C(p)\lambda^{\frac{1}{p}}$, see Proposition 8.4.

We recall that, by Proposition 6.6, there exist a one-to-one mapping,

$$(8.4) \quad T, T' \in \mathcal{BR}_E, T \rightsquigarrow T' \leftrightarrow \text{a square } K(T, T') \in \mathcal{K}_E,$$

between the family of (non-ordered) pairs of connected bridges $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$ and the family \mathcal{K}_E of well-separated squares $K(T, T') \in \mathcal{K}_E$ satisfying conditions (6.21) and (6.22).

The mapping (8.4) enables us to change our notation related to pairs of connected bridges. Since this mapping is one-to-one, we may consider the converse mapping which to every square $K \in \mathcal{K}_E$ assigns a pair of connected bridges $T_K, T'_K \in \mathcal{BR}_E, T_K \rightsquigarrow T'_K$. We denote the ends of these bridges by A_{T_K}, B_{T_K} and $A_{T'_K}, B_{T'_K}$ respectively. Since $T_K \rightsquigarrow T'_K$, we have

$$\{A^{[T_K]}, B^{[T_K]}\} \cap \{A^{[T'_K]}, B^{[T'_K]}\} \neq \emptyset,$$

i.e., the bridges T_K and T'_K have a common end. Thus the set of ends

$$\{A^{[T_K]}, B^{[T_K]}, A^{[T'_K]}, B^{[T'_K]}\}$$

consists of either three or two points.

Let us consider two cases.

The first case. $\{A^{[T_K]}, B^{[T_K]}\} \neq \{A^{[T'_K]}, B^{[T'_K]}\}$, i.e.,

$$\#\{A^{[T_K]}, B^{[T_K]}, A^{[T'_K]}, B^{[T'_K]}\} = 3.$$

In this case there exists the unique common end

$$(8.5) \quad C^{(K)} := \{A^{[T_K]}, B^{[T_K]}\} \cap \{A^{[T'_K]}, B^{[T'_K]}\}.$$

Let

$$(8.6) \quad A^{(K)} \quad \text{and} \quad B^{(K)} \quad \text{be two remaining points of the set of ends.}$$

Thus this set forms a triangle $\Delta(K) := \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ (true or degenerate).

Since T_K and T'_K are connected bridges, according to our definitions the side $[A^{(K)}, B^{(K)}]$ of the triangle $\Delta(K)$ (which is opposite to the vertex $C^{(K)}$) *is not the smallest side of* $\Delta(K)$.

Let α_K be the angle in $\Delta(K)$ corresponding to the vertex $C^{(K)}$. (In particular, $\alpha_K = 0$ whenever $\Delta(K)$ is a degenerate triangle.) Then, by (6.8),

$$(8.7) \quad |\sin \alpha_K| \sim \mathbf{c}_{\Delta(K)} \operatorname{diam} \Delta(K).$$

Recall that $\mathbf{c}_{\Delta(K)} = 1/R_{\Delta(K)}$ is the Menger curvature of the triangle $\Delta(K)$, see (1.1).

Also we recall that, by (6.21),

$$(8.8) \quad \operatorname{diam} \Delta(K) = \operatorname{diam}\{A^{[T_K]}, B^{[T_K]}, A^{[T'_K]}, B^{[T'_K]}\} \sim \operatorname{diam} K.$$

In turn, by (6.22),

$$(8.9) \quad \Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\} \subset \gamma K.$$

This inclusion and equivalence (8.8) imply the following useful inclusions

$$(8.10) \quad Q(T_K, T'_K) \subset \gamma K \quad \text{and} \quad K \subset \gamma Q(T_K, T'_K).$$

In all cases $\gamma > 0$ is an absolute constant. Recall that

$$Q(T_K, T'_K) = Q(A^{[T_K]}, D(T_K, T'_K))$$

where

$$D(T_K, T'_K) = \operatorname{diam}\{A^{[T_K]}, B^{[T_K]}, A^{[T'_K]}, B^{[T'_K]}\}$$

is defined in (7.16).

Note that, by (8.7) and (8.8),

$$(8.11) \quad |\sin \alpha_K| \sim \mathbf{c}_{\Delta(K)} \operatorname{diam} K = \frac{1}{R_{\Delta(K)}} \operatorname{diam} K.$$

The second case.

$$\#\{A^{[T_K]}, B^{[T_K]}, A^{[T'_K]}, B^{[T'_K]}\} = 2,$$

i.e., $\{A^{[T_K]}, B^{[T_K]}\} = \{A^{[T'_K]}, B^{[T'_K]}\}$. In this case we denote the ends of the bridges by $A^{(K)}$ and $B^{(K)}$ and put $C^{(K)} := A^{(K)}$. Then, by (6.21) and (6.22),

$$\|A^{(K)} - B^{(K)}\| = \|C^{(K)} - B^{(K)}\| \sim \operatorname{diam} K,$$

and, as in the first case,

$$\{A^{(K)}, B^{(K)}, C^{(K)}\} \subset \gamma K.$$

We turn to definitions of the measure μ_E on \mathbf{R}^2 and the mapping $\mathcal{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$. These definitions are similar to those given in Section 3, see (3.15), (3.16) and (3.17).

Let

$$(8.12) \quad \mathcal{C}_E := \{c_K : K \in \mathcal{K}_E\}$$

be the family of centers of all squares from \mathcal{K}_E . By μ_E we denote a discrete Borel measure with $\text{supp } \mu_E \subset \mathcal{C}_E$ such that

$$(8.13) \quad \mu_E(\{c_K\}) := \mathbf{c}_{\Delta(K)}^p |K| \quad \text{for every square } K \in \mathcal{K}_E.$$

Thus

$$(8.14) \quad \mu_E(S) = \sum \left\{ \mathbf{c}_{\Delta(K)}^p |K| : K \in \mathcal{K}_E, c_K \in S \right\}$$

for every set $S \subset \mathbf{R}^2$. We refer to μ_E as *the Menger curvature measure* generated by the set E .

Let us define the mapping $\mathcal{T}(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$. For every square $K \in \mathcal{K}_E$ we put

$$(8.15) \quad \mathcal{T}(c_K; f) := \begin{cases} \nabla P_{\Delta(K)}[f], & \Delta(K) \text{ is a true triangle,} \\ 0, & \Delta(K) \text{ is a degenerate triangle.} \end{cases}$$

For every $x \in \mathbf{R}^2 \setminus \mathcal{C}_E$ we set

$$(8.16) \quad \mathcal{T}(x; f) := 0.$$

Let $\vec{\Sigma} := \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$. Our aim is to show that $\|\mathcal{T}(f)\|_{\vec{\Sigma}} \leq C(p)\lambda$ provided the part (b) of the hypothesis of Theorem 7.1 holds. We present a proof of this statement in Proposition 8.4. The main ingredient of this proof is the following

Theorem 8.3 *Let $2 < p < \infty$ and let $\tau > 0$. Let μ be a non-trivial non-negative Borel measure on \mathbf{R}^2 . A mapping $\mathcal{V} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ belongs to the space $\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$ provided $\mathcal{V} \in \vec{L}_{p,loc}(\mathbf{R}^2; \mu)$ and there exists a constant $\lambda > 0$ such that the following statement is true for a certain absolute positive constant γ :*

Let \mathcal{Q} be a finite family of pairwise disjoint squares in \mathbf{R}^2 . Suppose that to each square $Q \in \mathcal{Q}$ we have arbitrarily assigned two squares $Q', Q'' \in \mathcal{Q}$ such that $Q' \cup Q'' \subset \gamma Q$ and

$$(8.17) \quad (\text{diam } Q')^{p-2} \mu(Q') + (\text{diam } Q'')^{p-2} \mu(Q'') \leq \tau.$$

Then the following inequality

$$\sum_{Q \in \mathcal{Q}} \left(\frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} \|\mathcal{V}(x) - \mathcal{V}(y)\|^p d\mu(x) d\mu(y) \leq \lambda$$

holds.

Furthermore, $\|\mathcal{V}\|_{\vec{\Sigma}} \leq C(p, \tau) \lambda^{\frac{1}{p}}$.

For scalar mappings, i.e., for functions on \mathbf{R}^2 , the proof of this theorem is given in [40], Theorem 6.1. We obtain the statement of Theorem 8.3 by applying this result to every component of the mapping \mathcal{V} .

We also note that it suffices to prove this theorem for a certain fixed value of the parameter τ , say for $\tau = 1$. Then the general case follows from this particular case by transition to a new measure $\tilde{\mu} = \frac{1}{\tau} \mu$.

Proposition 8.4 *Let μ_E and $\mathcal{T}(f)$ be the measure and the mapping defined by the formulas (8.13), (8.15) and (8.16). The mapping $\mathcal{T}(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ belongs to the space*

$$\vec{\Sigma} := \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$$

provided there exists a constant $\lambda > 0$ such that the condition (b) of Theorem 7.1 is satisfied.

Furthermore,

$$\|\mathcal{T}(f)\|_{\vec{\Sigma}} \leq C(p) \lambda^{\frac{1}{p}}.$$

Proof. By Theorem 8.3, it suffices to show that for every finite family \mathcal{Q} of pairwise disjoint squares in \mathbf{R}^2 and for any choice of squares $Q', Q'' \in \mathcal{Q}$ such that $Q' \cup Q'' \subset \gamma Q$ and

$$(8.18) \quad (\text{diam } Q')^{p-2} \mu_E(Q') + (\text{diam } Q'')^{p-2} \mu_E(Q'') \leq 1$$

the following inequality

$$I := \sum_{Q \in \mathcal{Q}} \left(\frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} \iint_{Q' \times Q''} \|\mathcal{T}(x; f) - \mathcal{T}(y; f)\|^p d\mu(x) d\mu(y) \leq \lambda$$

holds.

We note that, by (8.14), for every square $Q \in \mathcal{Q}$ we have

$$\mu_E(Q) = \sum \left\{ \mathbf{c}_{\Delta(K)}^p |K| : K \in \mathcal{K}, c_K \in Q \right\}$$

so that $\mu_E(Q) = \sigma_p(Q; \mathcal{K})$, see (1.5). Hence

$$\begin{aligned} & (\text{diam } Q')^{p-2} \mu_E(Q') + (\text{diam } Q'')^{p-2} \mu_E(Q'') \\ &= (\text{diam } Q')^{p-2} \sigma_p(Q'; \mathcal{K}) + (\text{diam } Q'')^{p-2} \sigma_p(Q''; \mathcal{K}) \end{aligned}$$

proving that inequality (8.18) is equivalent to inequality (7.2) from part (b) of Theorem 7.1.

In turn, by definitions (8.13), (8.15) and (8.16),

$$\begin{aligned} & \iint_{Q' \times Q''} \|\mathcal{T}(x; f) - \mathcal{T}(y; f)\|^p d\mu(x) d\mu(y) \\ &= \sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in Q'}} \sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in Q''}} \|\nabla P_{\Delta(K')}[f] - \nabla P_{\Delta(K'')}[f]\|^p \mathbf{c}_{\Delta(K')}^p |K'| \mathbf{c}_{\Delta(K'')}^p |K''| \\ &= S_p(f : Q', Q''; \mathcal{K}). \end{aligned}$$

Hence

$$I = \sum_{Q \in \mathcal{Q}} \left(\frac{\text{diam } Q' \text{ diam } Q''}{\text{diam } Q} \right)^{p-n} S_p(f : Q', Q''; \mathcal{K})$$

so that, by (7.3), $I \leq \lambda$ proving the proposition. \blacksquare

8.3. Pre-selections and selections of the set-valued mapping G_f . We turn to the last step of the proof of Theorem 7.1. Following to the scheme presented in Section 2, at this step we define a pre-selection $\tilde{g}(f) : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ and a selection $g(f) : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ of the set-valued mapping $G_f : \mathcal{BR}_E \rightarrow \text{Aff}(\mathbf{R}^2)$.

By Proposition 8.4, the mapping $\mathcal{T}(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ belongs to the space

$$\vec{\Sigma} := \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E),$$

and its norm in this space is bounded by $C(p)\lambda$. Thus there exist mappings

$$\mathcal{T}_1(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2 \quad \text{and} \quad \mathcal{T}_2(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$$

such that

$$\mathcal{T}(f) = \mathcal{T}_1(f) + \mathcal{T}_2(f)$$

and the following conditions are satisfied:

(i). The mapping $\mathcal{T}_1(f) \in \vec{L}_p^1(\mathbf{R}^2)$ and

$$(8.19) \quad \|\mathcal{T}_1(f)\|_{\vec{L}_p^1(\mathbf{R}^2)} \leq C(p)\lambda^{\frac{1}{p}};$$

(ii). The mapping $\mathcal{T}_2(f) \in \vec{L}_p(\mathbf{R}^2; \mu_E)$ and

$$(8.20) \quad \|\mathcal{T}_2(f)\|_{\vec{L}_p(\mathbf{R}^2; \mu_E)} \leq C(p)\lambda^{\frac{1}{p}}.$$

Now by $\tilde{g}(f) : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ we denote a mapping which to every bridge $T \in \mathcal{BR}_E$ with ends at points $A^{[T]}, B^{[T]} \in E$ assigns a vector

$$(8.21) \quad \tilde{g}(T; f) := \mathcal{T}_1(A^{[T]}; f).$$

We refer to the mapping $\tilde{g}(f) : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ as a *pre-selection* of the the set-valued mapping G_f .

Finally, we define the required selection $g(f) : \mathcal{BR}_E \rightarrow \mathbf{R}^2$ of the set-valued mapping $G_f : \mathcal{BR}_E \rightarrow \text{Aff}(\mathbf{R}^2)$ by the following formula:

$$(8.22) \quad g(T; f) := \text{Pr}(\tilde{g}(T; f); G_f(T)), \quad T \in \mathcal{BR}_E.$$

Here given a straight line $\ell \subset \mathbf{R}^2$ and a point $x \in \mathbf{R}^2$ by $\text{Pr}(x; \ell)$ we denote *the orthogonal projection of x onto ℓ* .

Our aim is to show that the selection $g(f)$ is a Sobolev-type mapping with respect to a certain non-negative $L_p(\mathbf{R}^2)$ -function $h(f)$ such that $\|h(f)_{L_p(\mathbf{R}^2)}\| \leq C(p)\lambda^{\frac{1}{p}}$. Hence, by Claim 8.1 (or by Proposition 7.6), we obtain the required estimate

$$(8.23) \quad \|f\|_{L_p^2(\mathbf{R}^2)|_E} \leq C(p)\lambda^{\frac{1}{p}}.$$

We construct the function $h(f)$ as a sum of three non-negative functions

$$h_1(f), h_2(f), h_3(f) \in L_p(\mathbf{R}^2).$$

Their definitions are motivated by inequalities (8.19), (8.20) and by part (i) of the hypothesis of Theorem 1.2 respectively.

We begin with a definition of the function $h_1(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$. We put

$$(8.24) \quad h_1(f) := \|\nabla \mathcal{T}_1(f)\|.$$

Here as usual for a vector function $\mathcal{T}_1(f)$ the sign ∇ means differentiation according to coordinates; thus

$$h_1(f) = \|(\nabla \mathcal{T}_1^{(1)}(f), \nabla \mathcal{T}_1^{(2)}(f))\|$$

provided $\mathcal{T}_1(f) = (\mathcal{T}_1^{(1)}(f), \mathcal{T}_1^{(2)}(f))$. By (8.19) and definition (3.12),

$$\begin{aligned} \|\mathcal{T}_1(f)\|_{\tilde{L}_p^1(\mathbf{R}^2)} &= \|\mathcal{T}_1^{(1)}(f)\|_{L_p^1(\mathbf{R}^2)} + \|\mathcal{T}_1^{(2)}(f)\|_{L_p^1(\mathbf{R}^2)} \\ &= \|\nabla \mathcal{T}_1^{(1)}(f)\|_{L_p(\mathbf{R}^2)} + \|\nabla \mathcal{T}_1^{(2)}(f)\|_{L_p(\mathbf{R}^2)} \\ &\leq C(p)\lambda^{\frac{1}{p}}. \end{aligned}$$

Hence

$$(8.25) \quad \|h_1(f)\|_{L_p(\mathbf{R}^2)} \leq C(p)\lambda^{\frac{1}{p}}.$$

Let us introduce the function $h_2(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$. By (8.20),

$$\|\mathcal{T}_2(f)\|_{\tilde{L}_p(\mathbf{R}^2; \mu_E)} = \|\mathcal{T}(f) - \mathcal{T}_1(f)\|_{\tilde{L}_p(\mathbf{R}^2; \mu_E)} \leq C(p)\lambda^{\frac{1}{p}}.$$

By definitions (8.13) and (8.15),

$$\begin{aligned} \|\mathcal{T}_2(f)\|_{\tilde{L}_p(\mathbf{R}^2; \mu_E)}^p &= \int_{\mathbf{R}^2} \|\mathcal{T}_2(x, f)\|^p d\mu_E(x) = \int_{\mathbf{R}^2} \|\mathcal{T}(x, f) - \mathcal{T}_1(x, f)\|^p d\mu_E(x) \\ &= \sum_{K \in \mathcal{K}} \|\mathcal{T}(c_K, f) - \mathcal{T}_1(c_K, f)\|^p \mathbf{c}_{\Delta(K)}^p |K| \\ &= \sum_{K \in \mathcal{K}} \|\nabla P_{\Delta(K)}[f] - \mathcal{T}_1(c_K, f)\|^p \mathbf{c}_{\Delta(K)}^p |K|. \end{aligned}$$

This inequality motivates us to define the function $h_2(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by the following formula

$$(8.26) \quad h_2(f) := \sum_{K \in \mathcal{K}} \|\nabla P_{\Delta(K)}[f] - \mathcal{T}_1(c_K, f)\| \mathbf{c}_{\Delta(K)} \chi_K.$$

Since the squares of the collection \mathcal{K} are pairwise disjoint,

$$\|h_2(f)\|_{L_p(\mathbf{R}^2)} = \sum_{K \in \mathcal{K}} \|\nabla P_{\Delta(K)}[f] - \mathcal{T}_1(c_K, f)\|^p \mathbf{c}_{\Delta(K)}^p |K|$$

proving that $\|h_2(f)\|_{L_p(\mathbf{R}^2)} = \|\mathcal{T}_2(f)\|_{\tilde{L}_p(\mathbf{R}^2; \mu_E)}$. Hence, by (8.20),

$$(8.27) \quad \|h_2(f)\|_{L_p(\mathbf{R}^2)} \leq C(p)\lambda^{\frac{1}{p}}.$$

We turn to a definition of the function $h_3(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$. We let \mathcal{K}_d denote a subfamily of the family \mathcal{K} consisting of squares $K \in \mathcal{K}$ such that the triangle $\Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ is *degenerate*. Thus $A^{(K)}, B^{(K)}, C^{(K)}$ are three different collinear points.

Let us denote the vertices $A^{(K)}, B^{(K)}, C^{(K)}$ by letters $z_1^{(K)}, z_2^{(K)}$ and $z_3^{(K)}$ in such a way that $z_2^{(K)} \in (z_1^{(K)}, z_3^{(K)})$. Since $A^{(K)}, B^{(K)}, C^{(K)} \in \gamma K$, see (8.9),

$$(8.28) \quad z_1^{(K)}, z_2^{(K)}, z_3^{(K)} \in \gamma K$$

as well.

We put

$$(8.29) \quad h_3(f) := \sum_{K \in \mathcal{K}_d} (\text{diam } K)^{-1} \left| \frac{f(z_1^{(K)}) - f(z_2^{(K)})}{\|z_1^{(K)} - z_2^{(K)}\|_2} - \frac{f(z_2^{(K)}) - f(z_3^{(K)})}{\|z_2^{(K)} - z_3^{(K)}\|_2} \right| \chi_K.$$

Recall that $\|\cdot\|_2$ denotes the Euclidean norm in \mathbf{R}^2 . Since the squares of the family \mathcal{K}_d are pairwise disjoint,

$$\|h_3(f)\|_{L_p(\mathbf{R}^2)}^p = \sum_{K \in \mathcal{K}_d} \left| \frac{f(z_1^{(K)}) - f(z_2^{(K)})}{\|z_1^{(K)} - z_2^{(K)}\|_2} - \frac{f(z_2^{(K)}) - f(z_3^{(K)})}{\|z_2^{(K)} - z_3^{(K)}\|_2} \right|^p (\text{diam } K)^{2-p}$$

so that, by (8.28) and (1.3),

$$(8.30) \quad \|h_3(f)\|_{L_p(\mathbf{R}^2)} \leq C(p)\lambda^{\frac{1}{p}}.$$

Finally we put

$$h(f) := h_1(f) + h_2(f) + h_3(f).$$

Then, by (8.25), (8.27), and (8.30),

$$\|h(f)\|_{L_p(\mathbf{R}^2)} \leq C(p)\lambda^{\frac{1}{p}}.$$

Proposition 8.5 *Let $2 < p < \infty$ and let $q := (p + 2)/2$. Let $f : E \rightarrow \mathbf{R}$ be a function satisfying the hypothesis of Theorem 7.1.*

The mapping $g(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by the formula (8.22) has the following properties:

(i). For every bridge $T \in \mathcal{BR}_E$ with ends $A^{[T]}, B^{[T]} \in E$ we have

$$\langle g(T; f), A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]});$$

(ii). For every pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \rightsquigarrow T'$, the following inequality

$$\|g(T; f) - g(T'; f)\| \leq C(p) \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h^q(z; f) dz \right)^{\frac{1}{q}}$$

holds. Here $\gamma > 0$ is an absolute constant.

We postpone a proof of the proposition to the end of this section. Here we only note that this proposition and Proposition 7.6 (see also Claim 8.1) immediately imply the required inequality (8.23).

The proof of the proposition relies on a series of auxiliary statements. First of them is the classical Sobolev-Poincaré inequality for vector functions, see e.g., [27].

Proposition 8.6 *Let $2 < q \leq p < \infty$ and let a mapping $\vec{F} \in \vec{L}_p^1(\mathbf{R}^2)$. Then for every square $Q \subset \mathbf{R}^2$ and every $x, y \in Q$ the following inequality*

$$\|\vec{F}(x) - \vec{F}(y)\| \leq C(q) \operatorname{diam} Q \left(\frac{1}{|Q|} \int_Q \|\vec{F}(z)\|^q dz \right)^{\frac{1}{q}}$$

holds.

The next auxiliary result is the following

Lemma 8.7 *For every pair of connected bridges $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$, the following inequality*

$$\|\tilde{g}(T; f) - \tilde{g}(T'; f)\| \leq C(p) \operatorname{diam} Q(T, T') \left(\frac{1}{|Q(T, T')|} \int_{Q(T, T')} h_1^q(z; f) dz \right)^{\frac{1}{q}}$$

holds.

Proof. We recall that

$$\tilde{g}(T; f) = \mathcal{T}_1(A^{[T]}; f) \quad \text{and} \quad \tilde{g}(T'; f) = \mathcal{T}_1(A^{[T']}; f)$$

where $\mathcal{T}_1(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a Sobolev $\vec{L}_p^1(\mathbf{R}^2)$ -mapping. We also recall that $h_1(f) = \|\nabla \mathcal{T}_1(f)\|$ and $Q(T, T') = Q(A^{[T]}, D(T, T'))$ with $D(T, T') = \operatorname{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}$.

Hence $A^{[T]}, A^{[T']} \in Q(T, T')$. It remains to apply the Sobolev-Poincaré inequality to the mapping $\mathcal{T}(f)$, the points $A^{[T]}, A^{[T']}$, and the square $Q(T, T')$, and the lemma follows. ■

Let us consider a pair of connected bridges $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$, such that their ends $A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}$ are not collinear points in \mathbf{R}^2 . Recall that in this case

$$\#\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} = 3$$

so that these points are vertices of a triangle. Let $K = K(T, T') \in \mathcal{K}$ be the square which we assign to the bridges T and T' so that $\Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ is the triangle with vertices in $\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\}$, see (8.5) and (8.6). We recall that $\operatorname{diam} \Delta(K) \sim \operatorname{diam} K$ and $\Delta(K) \subset \gamma K$ with an absolute constant γ , see (8.8) and (8.9).

The next lemma presents additional geometric properties of the triangle $\Delta(K)$ and its connections with the set-valued mapping G_f , see (8.1).

Lemma 8.8 *Let $K \in \mathcal{K}$ and let $\Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ be a triangle formed by a pair of connected bridges $T, T' \in \mathcal{BR}_E, T \leftrightarrow T'$ (thus $K = K(T, T')$).*

(i). If $\Delta(K)$ is a degenerate triangle, i.e., $A^{(K)}, B^{(K)}$ and $C^{(K)}$ are three different collinear points, then the straight lines $G_f(T)$ and $G_f(T')$ are parallel.

(ii). If $\Delta(K)$ is a true triangle, then $G_f(T)$ and $G_f(T')$ are not parallel. They intersect each other at the point

$$G_f(T) \cap G_f(T') = \nabla P_{\Delta(K)}[f].$$

Furthermore, the angle $\alpha \in (0, \pi/2)$ between these straight lines coincides with the angle α_K of the vertex $C^{(K)} = C(T, T')$ in the triangle $\Delta(K)$, and the following equivalences

$$(8.31) \quad \sin \alpha \sim \mathbf{c}_{\Delta(K)} \text{ diam } \Delta(K) \sim \mathbf{c}_{\Delta(K)} \text{ diam } Q(T, T')$$

hold. The constants of these equivalences are absolute.

Proof. The straight line $G_f(T)$ is defined by the equation

$$\langle z, A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]})$$

so that $\vec{n}_T := A^{[T]} - B^{[T]} \perp G_f(T)$, i.e., the vector \vec{n}_T is orthogonal to $G_f(T)$. The same is true for the bridge T' , i.e., $\vec{n}_{T'} := A^{[T']} - B^{[T']} \perp G_f(T')$.

Recall that T and T' are connected bridges and $\Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ is a triangle formed by the ends of T, T' , i.e., by the points $A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}$. We also recall that in this case $C^{(K)}$ is the (unique) common point of the sets $\{A^{[T]}, B^{[T]}\}$ and $\{A^{[T']}, B^{[T']}\}$, see (8.5). Without loss of generality we may assume that

$$\{A^{[T]}, B^{[T]}\} = \{A^{(K)}, C^{(K)}\} \quad \text{and} \quad \{A^{[T']}, B^{[T']}\} = \{B^{(K)}, C^{(K)}\}.$$

Hence $A^{(K)} - C^{(K)} = \pm n_T$ so that

$$(8.32) \quad A^{(K)} - C^{(K)} \perp G_f(T).$$

The same is true for the bridge T' , i.e.,

$$(8.33) \quad B^{(K)} - C^{(K)} \perp G_f(T').$$

Thus if $\Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ is a degenerate triangle, i.e., $A^{(K)}, B^{(K)}$ and $C^{(K)}$ lie on a certain straight line, then $(A^{(K)} - C^{(K)}) \parallel (B^{(K)} - C^{(K)})$. Combining this property with (8.32) and (8.33) we conclude that $G_f(T)$ and $G_f(T')$ are parallel as well. This proves part (i) of the lemma.

Prove (ii). Suppose that $\Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ is a true triangle. Then

$$(A^{(K)} - C^{(K)}) \not\parallel (B^{(K)} - C^{(K)})$$

proving that $G_f(T) \not\parallel G_f(T')$.

Let us note that the points on the straight line

$$\begin{aligned} G_f(T) &:= \{z \in \mathbf{R}^2 : \langle z, A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]})\} \\ &= \{z \in \mathbf{R}^2 : \langle z, A^{(K)} - C^{(K)} \rangle = f(A^{(K)}) - f(C^{(K)})\}. \end{aligned}$$

can be identified with *the gradients of polynomials* $P \in \mathcal{P}_1$ *which interpolate* f *at the points* $A^{[T]}$ *and* $B^{[T]}$, or equivalently at $A^{(K)}$ and $C^{(K)}$. In other words,

$$G_f(T) := \{\nabla P : P \in \mathcal{P}_1, P(A^{(K)}) = f(A^{(K)}), P(C^{(K)}) = f(C^{(K)})\}.$$

Analogously,

$$G_f(T') := \{\nabla P : P \in \mathcal{P}_1, P(B^{(K)}) = f(B^{(K)}), P(C^{(K)}) = f(C^{(K)})\}.$$

Thus the point $G_f(T) \cap G_f(T')$ can be identified with the gradient of the affine polynomial which interpolates f at the points $A^{(K)}, B^{(K)}$ and $C^{(K)}$, i.e., at the vertices of the triangle $\Delta(K)$. This shows that

$$G_f(T) \cap G_f(T') = \nabla P_{\Delta(K)}[f].$$

Let us prove equivalences in (8.31). By (8.32) and (8.33), the angle α between the straight lines $G_f(T)$ and $G_f(T')$ coincides with the angle α_K between the sides $[A^{(K)}, C^{(K)}]$ and $[B^{(K)}, C^{(K)}]$ of the triangle $\Delta(K)$, i.e., with the angle of the vertex $C^{(K)}$ in $\Delta(K)$.

Hence, by (8.11),

$$\sin \alpha = \sin \alpha_K \sim \mathbf{c}_{\Delta(K)} \text{ diam } K.$$

On the other hand, by (8.8), $\text{diam } \Delta(K) \sim \text{diam } K$, and

$$\text{diam } Q(T, T') = 2 \text{diam}\{A^{[T]}, B^{[T]}, A^{[T']}, B^{[T']}\} \sim \text{diam } K.$$

These equivalences imply the required equivalences in (8.31) proving the lemma. \blacksquare

Lemma 8.9 *Let $K \in \mathcal{K}$ and let $\Delta(K)$ be a true triangle formed by a pair of connected bridges $T, T' \in \mathcal{BR}_E$, $T \leftrightarrow T'$ (thus $K = K(T, T')$). Then*

$$\|\tilde{g}(T; f) - \nabla P_{\Delta(K)}[f]\| \mathbf{c}_{\Delta(K)} \leq C \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} (h_1 + h_2)^q(z; f) dz \right)^{\frac{1}{q}}$$

where $\gamma \geq 1$ is an absolute constant.

Proof. Let $K \in \mathcal{K}$. Since the squares of the family \mathcal{K} are pairwise disjoint, by (8.26),

$$h_2(x; f) = \|\nabla P_{\Delta(K)}[f] - \mathcal{T}_1(c_K, f)\| \mathbf{c}_{\Delta(K)} \quad \text{for every } x \in K.$$

Hence,

$$I_1 := \|\nabla P_{\Delta(K)}[f] - \mathcal{T}_1(c_K, f)\| \mathbf{c}_{\Delta(K)} = \left(\frac{1}{|K|} \int_K h_2^q(z; f) dz \right)^{\frac{1}{q}}$$

so that, by (8.10),

$$I_1 = \left(\frac{1}{|K|} \int_K h_2^q(z; f) dz \right)^{\frac{1}{q}} \leq C \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_2^q(z; f) dz \right)^{\frac{1}{q}}.$$

In turn, by the second inclusion in (8.10), the square $K \subset \gamma Q(T, T')$ so that $c_K, A^{[T]} \in \gamma Q(T, T')$. Applying the Sobolev-Poincaré inequality, see Proposition 8.6, to the mapping $\mathcal{T}_1(f)$, the points c_K and $A^{[T]}$, and the square $\gamma Q(T, T')$, we obtain

$$\begin{aligned} \|\mathcal{T}_1(c_K; f) - \tilde{g}(T; f)\| &= \|\mathcal{T}_1(c_K; f) - \mathcal{T}_1(A^{[T]}; f)\| \\ &\leq C \operatorname{diam} Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} \|\nabla \mathcal{T}_1(z; f)\|^q dz \right)^{\frac{1}{q}}. \end{aligned}$$

But, by (8.24), $h_1(f) = \|\nabla \mathcal{T}_1(f)\|$ so that

$$\|\mathcal{T}_1(c_K; f) - \tilde{g}(T; f)\| \leq C \operatorname{diam} Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_1(z; f)^q dz \right)^{\frac{1}{q}}.$$

Hence

$$\begin{aligned} I_2 &:= \|\mathcal{T}_1(c_K; f) - \tilde{g}(T; f)\| \mathbf{c}_{\Delta(K)} \\ &\leq C \mathbf{c}_{\Delta(K)} \operatorname{diam} Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_1(z; f)^q dz \right)^{\frac{1}{q}}. \end{aligned}$$

Note that, by (8.8) and (8.11),

$$\mathbf{c}_{\Delta(K)} \operatorname{diam} Q(T, T') \leq C \mathbf{c}_{\Delta(K)} \operatorname{diam} K \leq C |\sin \alpha_K|$$

so that

$$\begin{aligned} I_2 &\leq C |\sin \alpha_K| \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_1(z; f)^q dz \right)^{\frac{1}{q}} \\ &\leq C \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_1(z; f)^q dz \right)^{\frac{1}{q}}. \end{aligned}$$

Finally, since $h_1(f)$ and $h_2(f)$ are non-negative, we have

$$\begin{aligned} \|\tilde{g}(T; f) - \nabla P_{\Delta(K)}[f]\| \mathbf{c}_{\Delta(K)} &\leq I_1 + I_2 \leq C \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_2(z; f)^q dz \right)^{\frac{1}{q}} \\ &\quad + \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_1(z; f)^q dz \right)^{\frac{1}{q}} \\ &\leq C \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} (h_1 + h_2)^q(z; f) dz \right)^{\frac{1}{q}}. \end{aligned}$$

The lemma is proved. ■

Given subsets $A_1, A_2 \subset \mathbf{R}^2$ we let $\text{dist}_2(A_1, A_2)$ denote the Euclidean distance between A_1 and A_2 :

$$\text{dist}_2(A_1, A_2) = \inf\{\|a_1 - a_2\|_2 : a_1 \in A_1, a_2 \in A_2\}.$$

Recall that by $\|\cdot\|_2$ we denote the Euclidean norm in \mathbf{R}^2 .

Lemma 8.10 *Let $K \in \mathcal{K}$ and let $\Delta(K)$ be a degenerate triangle formed by a pair of connected bridges $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$. Then the straight lines $G_f(T)$ and $G_f(T')$ are parallel. Furthermore,*

$$\text{dist}_2(G_f(T), G_f(T')) \leq C \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_3^q(z; f) dz \right)^{\frac{1}{q}}.$$

Here $\gamma \geq 1$ is an absolute constant.

Proof. We recall that $A^{(K)}, B^{(K)}$ and $C^{(K)}$ are three different collinear points in \mathbf{R}^2 provided $\Delta(K) = \Delta\{A^{(K)}, B^{(K)}, C^{(K)}\}$ is a degenerate triangle. By part (i) of Lemma 8.8, in this case $G_f(T)$ and $G_f(T')$ are parallel straight lines.

Let us denote the vertices $A^{(K)}, B^{(K)}, C^{(K)}$ by letters $z_1^{(K)}, z_2^{(K)}$ and $z_3^{(K)}$ in such a way that $z_2^{(K)} \in (z_1^{(K)}, z_3^{(K)})$. Then the straight lines $G_f(T)$ and $G_f(T')$ are determined by the equations

$$\langle z, z_1^{(K)} - z_2^{(K)} \rangle = f(z_1^{(K)}) - f(z_2^{(K)})$$

and

$$\langle z, z_2^{(K)} - z_3^{(K)} \rangle = f(z_2^{(K)}) - f(z_3^{(K)})$$

respectively. The Euclidean distance between these parallel straight lines is equal to

$$\text{dist}_2(G_f(T), G_f(T')) = \left| \frac{f(z_1^{(K)}) - f(z_2^{(K)})}{\|z_1^{(K)} - z_2^{(K)}\|_2} - \frac{f(z_2^{(K)}) - f(z_3^{(K)})}{\|z_2^{(K)} - z_3^{(K)}\|_2} \right|.$$

Hence, by (8.29),

$$h_3(x, f) = \text{dist}_2(G_f(T), G_f(T')) / \text{diam } K \quad \text{for every } x \in K.$$

Raising this equality to the power q and then integrating on the square K we obtain

$$\text{dist}_2(G_f(T), G_f(T')) = \text{diam } K \left(\frac{1}{|K|} \int_K h_3^q(z; f) dz \right)^{\frac{1}{q}}.$$

It remains to note that, by (8.10), $K \subset \gamma Q(T, T')$ and $\text{diam } K \sim \text{diam } Q(T, T')$, so that

$$\text{dist}(G_f(T), G_f(T')) \leq C \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_3^q(z; f) dz \right)^{\frac{1}{q}}$$

proving the lemma. ■

Lemma 8.11 *Let ℓ_1 and ℓ_2 be two non-parallel straight lines in \mathbf{R}^2 intersecting at a point $a \in \mathbf{R}^2$. Let $\alpha \in (0, \pi/2)$ be the angle between these straight lines.*

Then for every $x, y \in \mathbf{R}^2$ the following inequality

$$(8.34) \quad \|\Pr(x; \ell_1) - \Pr(y; \ell_2)\|_2 \leq \|x - y\|_2 + 2\sqrt{2} \sin \alpha \|x - a\|_2$$

holds.

Proof. Without loss of generality we may assume that $a = 0$, i.e., ℓ_1 and ℓ_2 are one dimensional linear subspaces of \mathbf{R}^2 . We may also assume that $\ell_1 = A\ell_2$ where $A : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a rotation operator by the angle α . Note that for every $z \in \mathbf{R}^2$ and every linear subspace $\ell \subset \mathbf{R}^2$, $\dim \ell = 1$, we have $\Pr(Az; A\ell) = A(\Pr(z; \ell))$ so that

$$(8.35) \quad \Pr(Ax; \ell_1) = A(\Pr(x; \ell_2)).$$

Now we have

$$\|\Pr(x; \ell_1) - \Pr(y; \ell_2)\|_2 \leq \|\Pr(x; \ell_1) - \Pr(x; \ell_2)\|_2 + \|\Pr(x; \ell_2) - \Pr(y; \ell_2)\|_2$$

so that

$$(8.36) \quad \|\Pr(x; \ell_1) - \Pr(y; \ell_2)\|_2 \leq \|\Pr(x; \ell_1) - \Pr(x; \ell_2)\|_2 + \|x - y\|_2.$$

Let us estimate the quantity

$$I := \|\Pr(x; \ell_1) - \Pr(x; \ell_2)\|_2.$$

We have

$$\begin{aligned} I &\leq \|\Pr(x; \ell_1) - \Pr(Ax; \ell_1)\|_2 + \|\Pr(Ax; \ell_1) - \Pr(x; \ell_2)\|_2 \\ &\leq \|x - Ax\|_2 + \|\Pr(Ax; \ell_1) - \Pr(x; \ell_2)\|_2. \end{aligned}$$

Hence, by (8.35),

$$(8.37) \quad I \leq \|x - Ax\|_2 + \|A(\Pr(x; \ell_2)) - \Pr(x; \ell_2)\|_2.$$

Let Id be the identity operator on \mathbf{R}^2 and let $B := \text{Id} - A$. Then, by (8.37),

$$I \leq \|Bx\|_2 + \|B(\Pr(x; \ell_2))\|_2.$$

Since B commutes with rotations and $\|\Pr(x; \ell_2)\|_2 \leq \|x\|_2$, we have

$$\|B(\Pr(x; \ell_2))\|_2 \leq \|Bx\|_2$$

proving that $I \leq 2\|Bx\|_2$.

Also, using a rotation, we obtain that

$$\|Bx\|_2 = \|Be_1\|_2 \|x\|_2 \quad \text{where } e_1 = (1, 0).$$

Simple calculation shows that $\|Be_1\|_2 = 2 \sin(\alpha/2)$. But $2 \sin(\alpha/2) \leq \sqrt{2} \sin \alpha$ for $\alpha \in (0, \pi/2)$ so that $\|Be_1\|_2 \leq \sqrt{2} \sin \alpha$. Hence

$$I \leq 2\|Bx\|_2 \leq 2\sqrt{2} \sin \alpha \|x\|_2 = 2\sqrt{2} \sin \alpha \|x - a\|_2.$$

Combining this estimate with (8.36) we obtain the required inequality (8.34). \blacksquare

We are in a position to prove Proposition 8.5.

Proof of Proposition 8.5. Part (i) of the proposition is trivial because for each $T \in \mathcal{BR}_E$ the point $g(T; f) = \Pr(\tilde{g}(T; f); G_f(T))$ belongs to the straight line $G_f(T)$ which is determined by the equation $\langle z, A^{[T]} - B^{[T]} \rangle = f(A^{[T]}) - f(B^{[T]})$.

Let us prove part (ii) of the proposition. Let $T, T' \in \mathcal{BR}_E, T \rightsquigarrow T'$, be a pair of connected bridges. Let $K = K(T, T')$ be the square from the family \mathcal{K} corresponding to the bridges T and T' . Consider three cases.

The first case: the triangle $\Delta(K)$ formed by the ends of the bridges T and T' is a true triangle.

By Lemma 8.8, the straight lines $G_f(T)$ and $G_f(T')$ are non-parallel and

$$G_f(T) \cap G_f(T') = \nabla P_{\Delta(K)}[f].$$

Let $\alpha \in (0, \pi/2)$ be the angle between these straight lines. Then, by Lemma 8.11,

$$\begin{aligned} \|g(T; f) - g(T'; f)\| &= \|\Pr(\tilde{g}(T; f); G_f(T)) - \Pr(\tilde{g}(T'; f); G_f(T'))\| \\ &\leq C\{\|\tilde{g}(T; f) - \tilde{g}(T'; f)\| + \|\tilde{g}(T; f) - \nabla P_{\Delta(K)}[f]\| \sin \alpha\} \\ &= C\{I_1 + I_2\}. \end{aligned}$$

By Lemma 8.7,

$$I_1 := \|\tilde{g}(T; f) - \tilde{g}(T'; f)\| \leq C(p) \operatorname{diam} Q(T, T') \left(\frac{1}{|Q(T, T')|} \int_{Q(T, T')} h_1^q(z; f) dz \right)^{\frac{1}{q}}.$$

Let us estimate the quantity

$$I_2 := \|\tilde{g}(T; f) - \nabla P_{\Delta(K)}[f]\| \sin \alpha.$$

By equivalence (8.31),

$$\sin \alpha \sim \mathbf{c}_{\Delta(K)} \operatorname{diam} Q(T, T')$$

so that

$$I_2 \leq C \|\tilde{g}(T; f) - \nabla P_{\Delta(K)}[f]\| \mathbf{c}_{\Delta(K)} \operatorname{diam} Q(T, T').$$

Hence, by Lemma 8.9,

$$I_2 \leq C \operatorname{diam} Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} (h_1 + h_2)^q(z; f) dz \right)^{\frac{1}{q}}$$

with some absolute constant $\gamma \geq 1$.

Summarizing the estimates for I_1 and I_2 we obtain

$$\begin{aligned} \|g(T; f) - g(T'; f)\| &\leq C\{I_1 + I_2\} \\ &\leq C \operatorname{diam} Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} (h_1 + h_2)^q(z; f) dz \right)^{\frac{1}{q}}. \end{aligned}$$

The second case: the triangle $\Delta(K)$ is a degenerate triangle.

By Lemma 8.10, $G_f(T)$ and $G_f(T')$ are parallel straight lines so that

$$\begin{aligned}
\|g(T; f) - g(T'; f)\| &\leq \|g(T; f) - g(T'; f)\|_2 \\
&= \|\Pr(\tilde{g}(T; f); G_f(T)) - \Pr(\tilde{g}(T'; f); G_f(T'))\|_2 \\
&\leq \|\Pr(\tilde{g}(T; f); G_f(T)) - \Pr(\tilde{g}(T'; f); G_f(T))\|_2 \\
&+ \|\Pr(\tilde{g}(T'; f); G_f(T)) - \Pr(\tilde{g}(T'; f); G_f(T'))\|_2 \\
&\leq \|\tilde{g}(T; f) - \tilde{g}(T'; f)\|_2 + \text{dist}_2(G_f(T), G_f(T')).
\end{aligned}$$

By Lemma 8.7,

$$\|\tilde{g}(T; f) - \tilde{g}(T'; f)\| \leq C \text{diam } Q(T, T') \left(\frac{1}{|Q(T, T')|} \int_{Q(T, T')} h_1^q(z; f) dz \right)^{\frac{1}{q}}.$$

In turn, by Lemma 8.10,

$$\text{dist}(G_f(T), G_f(T'))_2 \leq C \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} h_3^q(z; f) dz \right)^{\frac{1}{q}}.$$

Hence

$$\|g(T; f) - g(T'; f)\| \leq C \text{diam } Q(T, T') \left(\frac{1}{|\gamma Q(T, T')|} \int_{\gamma Q(T, T')} (h_1 + h_3)^q(z; f) dz \right)^{\frac{1}{q}}.$$

The third case: the set of the ends of the bridges T and T' is a two point set.

In this case $\{A^{[T]}, B^{[T]}\} = \{A^{[T']}, B^{[T']}\}$ so that, by definition (8.1), $G_f(T) = G_f(T')$.

Hence

$$\begin{aligned}
\|g(T; f) - g(T'; f)\| &\leq \|g(T; f) - g(T'; f)\|_2 \\
&= \|\Pr(\tilde{g}(T; f); G_f(T)) - \Pr(\tilde{g}(T'; f); G_f(T'))\|_2 \\
&\leq \|\tilde{g}(T; f) - \tilde{g}(T'; f)\|_2
\end{aligned}$$

so that, by Lemma 8.7,

$$\|g(T; f) - g(T'; f)\| \leq C \text{diam } Q(T, T') \left(\frac{1}{|Q(T, T')|} \int_{Q(T, T')} h_1^q(z; f) dz \right)^{\frac{1}{q}}.$$

We have shown that in each of three possible cases the condition (ii) of the proposition holds. This proves part (ii) and finishes the proof of the proposition. ■

Theorem 7.1 is completely proved. ■

9. Algorithms of an almost optimal decomposition of $\vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$.

We recall that in Subsection 8.2 we have defined a discrete Borel measure μ_E (which we call the Menger curvature measure) and a mapping $\mathcal{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, see formulas (8.13) - (8.16).

Basing on Theorem 8.3 and Proposition 8.4, in Subsection 8.3 we claim the existence of the mappings

$$\mathcal{T}_1(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2 \quad \text{and} \quad \mathcal{T}_2(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$$

such that $\mathcal{T}(f) = \mathcal{T}_1(f) + \mathcal{T}_2(f)$ with the norms satisfying inequalities (8.19) and (8.20).

Note that the mapping \mathcal{T}_1 is one of the main ingredients of our extension construction because this operator determines the pre-selection $\tilde{g}(f)$ given by formula (8.21).

Let $\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$. In this section we describe a constructive algorithm which to every mapping $\mathcal{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ assigns a mapping $\mathcal{T}_1 : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ possessing the following properties: the mapping \mathcal{T}_1 *depends linearly on* \mathcal{T} and the following inequalities

$$\|\mathcal{T}_1\|_{\vec{L}_p^1(\mathbf{R}^2)} \leq C(p)\|\mathcal{T}\|_{\vec{\Sigma}} \quad \text{and} \quad \|\mathcal{T} - \mathcal{T}_1\|_{\vec{L}_p(\mathbf{R}^2; \mu_E)} \leq C(p)\|\mathcal{T}\|_{\vec{\Sigma}}$$

hold. This enables us to put $\mathcal{T}_1(f) = (\mathcal{T}(f))_1$.

The algorithm is based on an approach suggested in the author's paper [40]. In the next two subsections we describe two main parts of this algorithm.

9.1 “Important Squares” and concentration of the Menger curvature measure.

Let \mathcal{K}_E be the family of well-separated squares constructed in Proposition 6.6, and let μ_E be the Menger curvature measure defined in Subsection 8.2, see (8.13) and (8.14). Recall that

$$\mathcal{C}_E = \{c_K : K \in \mathcal{K}_E\}$$

denotes the family of centers of all squares from \mathcal{K}_E .

We also recall that for each square $K \in \mathcal{K}_E$ the triangle $\Delta(K)$ has the following properties: $\Delta(K) \subset \gamma K$ and $\text{diam } \Delta(K) \sim \text{diam } K$ where γ and the constants of the last equivalence are absolute. Hence

$$\mathbf{c}_{\Delta(K)} = \frac{1}{R_{\Delta(K)}} \leq \frac{C}{\text{diam } \Delta(K)} \leq \frac{C}{r_K}$$

with an absolute constant C . Then, by (8.13),

$$\mu_E(K) = \mathbf{c}_{\Delta(K)}^p |K| \leq \frac{C^p (2r_K)^2}{r_K^p} = 4C^p r_K^{2-p}$$

so that

$$(9.1) \quad \mu_E(K) \leq \eta r_K^{2-p}$$

where $\eta := 4C^p$.

Let us consider a function

$$S_K(r) := \mu_E(Q(c_K, r)), \quad r \geq 0.$$

This function is non-decreasing non-negative and right continuous on $[0, \infty)$. Furthermore, by (9.1), $S_K(r) \leq \eta r_K^{2-p}$.

Let $V(r) := \eta r^{2-p}$, $r > 0$. Then

$$(9.2) \quad S_K(r_K) \leq V(r_K).$$

Since $p > 2$, the function $V = V(r)$ is *strictly decreasing*. These properties of the functions $S_K(r)$ and $V(r)$ imply the existence of a unique number $R_K \in (0, \infty)$ such that

$$(9.3) \quad \mu_E(Q(c_K, r)) > \eta R_K^{2-p} \quad \text{for every } r > R_K,$$

and

$$(9.4) \quad \mu_E(Q(c_K, r)) < \eta R_K^{2-p} \quad \text{for every } r < R_K.$$

Note that, by (9.2),

$$(9.5) \quad R_K \geq r_K.$$

Also note that, since the function S_K is right continuous on $(0, \infty)$, by (9.3),

$$\mu_E(Q(c_K, R_K)) \geq \eta R_K^{2-p}.$$

Note one more property of the function $K \rightarrow R_K$ proven in [40], Lemma 3.4: for every $K, K' \in \mathcal{K}_E$ the following inequality

$$|R_K - R_{K'}| \leq \|c_K - c_{K'}\|$$

holds.

Given a square $K \in \mathcal{K}_E$ we let \tilde{K} denote a square

$$(9.6) \quad \tilde{K} := Q(c_K, R_K).$$

Note that, by (9.5), $K \subset \tilde{K}$ for every $K \in \mathcal{K}_E$. Let

$$\tilde{\mathcal{K}}_E := \{\tilde{K} : K \in \mathcal{K}_E\}.$$

Our aim at this stage of the algorithm is to construct a subfamily \mathcal{Q}_E of the family \mathcal{K}_E consisting of well separated squares which provides a certain *net* in the collection \mathcal{K}_E . This means that for every square $\tilde{K} \in \tilde{\mathcal{K}}_E$ its fixed dilation (say, by a factor of 12) contains a square from the family \mathcal{Q}_E .

The existence of such a family \mathcal{Q}_E is proven in [40], Proposition 3.5. See also [40], Subsection 6.3.

Proposition 9.1 *There exists a subfamily \mathcal{Q}_E of the family $\tilde{\mathcal{K}}_E$ such that the following conditions are satisfied:*

(i). *The squares of the family*

$$6\mathcal{Q}_E := \{6Q : Q \in \mathcal{Q}_E\}$$

are pairwise disjoint;

(ii). *For each square $K \in \tilde{\mathcal{K}}_E$ there exists a square $\hat{K} \in \mathcal{Q}_E$ such that*

$$\hat{K} \subset 12K.$$

Proof. In [40] we prove a general result of such a kind for an *arbitrary* (not necessarily finite) family of cubes in \mathbf{R}^n . Here for the case of a *finite* family of squares $\tilde{\mathcal{K}}_E$ we present a short proof of this result due to V. Dol'nikov.

Let $\mathcal{A} := 6\tilde{\mathcal{K}}_E$. Let K_1 be a square of the minimal diameter among all the squares of the family $\mathcal{A}_1 := \mathcal{A}$. By G_1 we denote all squares of \mathcal{A}_1 which intersect K_1 .

We put $\mathcal{A}_2 := \mathcal{A}_1 \setminus G_1$. If $\mathcal{A}_2 = \emptyset$ we stop and put $\mathcal{B} = \{K_1\}$. If $\mathcal{A}_2 \neq \emptyset$, by K_2 we denote a square of the minimal diameter among all the squares of the family \mathcal{A}_2 . We continue this procedure. Since \mathcal{A} is finite, this process will stop on a certain (finite) step m .

As a result we obtain a finite collection of pairwise disjoint squares $\mathcal{B} = \{K_1, \dots, K_m\}$ and a partition $\{G_1, \dots, G_m\}$ of \mathcal{A} such that for each $1 \leq i \leq m$ the following conditions are satisfied: the square $K_i \in G_i$, $K_i \cap Q \neq \emptyset$, and $\text{diam } K_i \leq \text{diam } Q$ for every $Q \in G_i$.

Thus \mathcal{B} is a collection of pairwise disjoint squares possessing the following property: for each square $Q \in \mathcal{A}$ there exists a square $\bar{Q} \in \mathcal{B}$ such that $\bar{Q} \cap Q \neq \emptyset$ and $\text{diam } \bar{Q} \leq \text{diam } Q$. Clearly, $\bar{Q} \subset 2Q$.

It remains to put

$$\mathcal{Q}_E := \frac{1}{6}\mathcal{B} = \{\frac{1}{6}Q : Q \in \mathcal{B}\}.$$

Then \mathcal{Q}_E is a subfamily $\tilde{\mathcal{K}}_E$ satisfying conditions (i) and (ii) of the proposition. In fact, since the squares of the family $\mathcal{B} = 6\mathcal{Q}_E$ are pairwise disjoint, the condition (i) holds. To prove the condition (ii) fix a square $K \in \tilde{\mathcal{K}}_E$. Then the square $Q = 6K \in \mathcal{A}$ so that there exists a square $\bar{Q} \in \mathcal{B}$ such that $\bar{Q} \subset 2Q = 12K$.

Then the square $\hat{K} := \frac{1}{6}\bar{Q} \in \frac{1}{6}\mathcal{B} = \mathcal{Q}_E$, and $\hat{K} \subset \bar{Q} \subset 12K$ proving the proposition. \blacksquare

We refer to each square Q which belong to the family \mathcal{Q}_E as an “important” square. As we will see in the next subsection the squares of the family \mathcal{Q}_E accumulate all information which is necessary for an almost optimal decomposition of every mapping $\mathcal{T} \in \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$.

Let us present main properties of the “important” squares. For the proof of these properties we refer the reader to the work [40], Section 3.

Let

$$S_E := \bigcup_{Q \in \mathcal{Q}_E} Q.$$

- For every $Q \in \mathcal{Q}_E$

$$\mu_E(Q) \sim \mu_E(5Q) \sim (\text{diam } Q)^{2-p}$$

with constants of the equivalences depending only on p .

- Let $x \in \mathbf{R}^2$ and let $r(x)$ be a (unique) positive number such that

$$\mu_E(Q(x, r)) > \eta r(x)^{2-p} \quad \text{for every } r > r(x),$$

and

$$\mu_E(Q(x, r)) < \eta r(x)^{2-p} \quad \text{for every } r < r(x).$$

Clearly, by (9.3) and (9.4), $r(c_K) = R_K$ for every square $K \in \mathcal{K}_E$ so that $\tilde{\mathcal{K}}_E \subset \tilde{\mathcal{K}}$ where

$$\tilde{\mathcal{K}} := \{Q(x, r(x)) : x \in \mathbf{R}^2\}.$$

In spite of the fact that the family $\tilde{\mathcal{K}}$ contains $\tilde{\mathcal{K}}_E$, *the same family \mathcal{Q}_E provides a similar “net” in $\tilde{\mathcal{K}}$ as well.* In other words, there exists a constant $\gamma > 0$ such that for every $x \in \mathbf{R}^2$ there exists a square $K^{(x)} \in \mathcal{Q}_E$ for which the following inclusion

$$K^{(x)} \subset \gamma Q(x, r(x))$$

holds.

• **“Important Squares” as sets of concentration of the Menger curvature measure.** Let $Q \in \mathcal{Q}_E$ and let $\theta > 0$. Let K be a square in \mathbf{R}^2 such that $K \supset Q$ and

$$\text{diam } K \leq \theta \text{dist}(K, S_E \setminus Q).$$

Then

$$\mu_E(K) \leq C \mu_E(Q)$$

where C is a constant depending only on p and θ .

This property easily follows from the next result proven in [40], Lemma 3.7: for every $\theta > 0$ and every square $Q \subset \mathbf{R}^2$ such that $\text{diam } Q \leq \theta \text{dist}(Q, S_E)$ the following inequality

$$\mu_E(Q) \leq C(\theta, p)(\text{diam } Q)^{2-p}$$

holds.

9.2 Whitney-type extensions from “important squares”.

We are in a position to construct an almost optimal decomposition of a mapping

$$\mathcal{T} \in \vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E).$$

We let \mathcal{C} denote the set of centers of all squares from the collection \mathcal{Q}_E :

$$\mathcal{C} := \{c_Q : Q \in \mathcal{Q}_E\}.$$

Thus $\mathcal{C} \subset \mathcal{C}_E$.

Given $x \in \mathcal{C}$ we let $Q^{(x)}$ denote the (unique) square from \mathcal{Q}_E with center at the point x . Hence $\mathcal{Q}_E = \{Q^{(x)} : x \in \mathcal{C}\}$.

The μ_E -measure concentration property of “important squares” motivates us to determine the component $\mathcal{T}_1 \in \vec{L}_p^1(\mathbf{R}^2)$ of an almost optimal decomposition of \mathcal{T} using the following Whitney-type extension construction:

Step 1. We define a new mapping $\tilde{\mathcal{T}} : \mathcal{C} \rightarrow \mathbf{R}^2$ by the formula

$$\tilde{\mathcal{T}}(x) = \frac{1}{\mu_E(Q^{(x)})} \int_{Q^{(x)}} \mathcal{T} d\mu_E, \quad x \in \mathcal{C}.$$

Step 2. We extend the mapping $\tilde{\mathcal{T}}$ from the set \mathcal{C} to all of \mathbf{R}^2 using the classical Whitney Extension Method. More specifically, let $W_{\mathcal{C}}$ be a Whitney covering of the open set $\mathbf{R}^2 \setminus \mathcal{C}$, and let $\{\psi_Q : Q \in W_{\mathcal{C}}\}$ be a smooth partition of unity subordinated to $W_{\mathcal{C}}$, see Section 4, Theorem 4.1, and Section 7, Lemma 7.2.

Let $\eta > 1$. Given a square $Q \in W_{\mathcal{C}}$ we let A_Q denote a point which belongs to the set $(\eta Q) \cap \mathcal{C}$. (In particular, one can choose A_Q to be a point nearest to Q on the set \mathcal{C} .)

Then we define the mapping \mathcal{T}_1 by the Whitney formula:

$$(9.7) \quad \mathcal{T}_1(x) := \begin{cases} \tilde{\mathcal{T}}(x), & x \in \mathcal{C}, \\ \sum_{Q \in W_{\mathcal{C}}} \psi_Q(x) \tilde{\mathcal{T}}(A_Q), & x \in \mathbf{R}^2 \setminus \mathcal{C}. \end{cases}$$

The next theorem is a particular case of a general result proven in [40].

Theorem 9.2 *Let $2 < p < \infty$ and let a mapping $\mathcal{T} \in \vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu_E)$. Then the mapping $\mathcal{T}_1 \in \vec{L}_p^1(\mathbf{R}^2)$ and the mapping $\mathcal{T} - \mathcal{T}_1 \in \vec{L}_p(\mathbf{R}^2; \mu_E)$. Furthermore, the following inequalities*

$$\|\mathcal{T}_1\|_{\vec{L}_p^1(\mathbf{R}^2)} \leq C \|\mathcal{T}\|_{\vec{\Sigma}}, \quad \|\mathcal{T} - \mathcal{T}_1\|_{\vec{L}_p(\mathbf{R}^2; \mu_E)} \leq C \|\mathcal{T}\|_{\vec{\Sigma}}$$

hold. Here C is a constant depending only on p and η .

Remark 9.3 In [40], Section 6, we extend the mapping $\tilde{\mathcal{T}}$ to a mapping $\mathcal{T}_1 \in \vec{L}_p^1(\mathbf{R}^2)$ using a *lacunary modification of the Whitney extension method*. This modification relies on the same ideas as our modification of the Whitney construction given in Section 7.

In particular, let $\mathcal{L}_{\mathcal{C}}$ be the family of all lacunas of Whitney squares of the set \mathcal{C} , see Section 4. By Proposition 5.3, there exists a “projection” which to every lacuna $L \in \mathcal{L}_{\mathcal{C}}$ assigns a point $\mathcal{PR}(L) \in \mathcal{C}$ such that

$$(9.8) \quad \mathcal{PR}(L) \in (\gamma Q_L) \cap \mathcal{C}.$$

Furthermore, the mapping $L \rightarrow \mathcal{PR}(L)$ is an “almost” one-to-one mapping, i.e., each point $a \in \mathbf{R}^2$ has at most N pre-images:

$$(9.9) \quad \#\{L \in \mathcal{L}_{\mathcal{C}} : \mathcal{PR}(L) = a\} \leq N \quad \text{for every } a \in \mathbf{R}^2.$$

Here γ and N are absolute constants.

Given a lacuna $L \in \mathcal{L}_{\mathcal{C}}$ we put

$$A_Q := \mathcal{PR}(L) \quad \text{for all squares } Q \in L.$$

Then we define a new mapping $\mathcal{T}_1^{(\ell)}$ by the same formula (9.7).

By Proposition 9.2,

$$\|\mathcal{T}_1^{(\ell)}\|_{\vec{L}_p^1(\mathbf{R}^2)} \leq C \|\mathcal{T}\|_{\vec{\Sigma}}, \quad \|\mathcal{T} - \mathcal{T}_1^{(\ell)}\|_{\vec{L}_p(\mathbf{R}^2; \mu_E)} \leq C \|\mathcal{T}\|_{\vec{\Sigma}}.$$

In the forthcoming paper [42] we will show that this modification provides a continuous linear extension operator with a rather simple and “nice” structure. \triangleleft

10. The extension algorithm and its linearity.

In this section we describe main stages of the algorithm which provides an almost optimal extension of every function from the trace space $L_p^2(\mathbf{R}^2)|_E$ whenever $E \subset \mathbf{R}^2$ is an arbitrary finite set and $p > 2$. An optimality of this algorithm follows from the proof of Theorem 1.2. We will see that this algorithm provides a *continuous linear extension operator* for the trace space $L_p^2(\mathbf{R}^2)|_E$.

The algorithm consists of two main parts which we call “Pre-work” and “Main Algorithm”.

In the part “Pre-work” we introduce a series of geometrical objects (lacunae, bridges, families of triangles, measures, etc.) which we use later on in our extension construction. Note that all these objects are determined only by a geometrical structure of the set E and does not depend on values of functions defined on E .

In the part “Main Algorithm” we fix a function $f : E \rightarrow \mathbf{R}$. Its values on the set E we consider as input data of our algorithm. Then we show how these input data transform into the values of an almost optimal extension $F \in L_p^2(\mathbf{R}^2)$ of the function f . At every step of the algorithm we control its “linearity”, i.e., linear dependence of all elements of our construction on the data.

In this section our aim is only to present more or less detailed description of the extension algorithm. In this paper we do not estimate its complexity and the order of magnitude of the number of elementary operation which is necessary for computation of the extension $F \in L_p^2(\mathbf{R}^2)$. We will study these and other problems related to the optimality of the algorithm in the forthcoming paper [42].

10.1. Pre-work: lacunae, bridges, and measure concentration squares.

Step 1: Whitney squares and a partition of unity. We fix a family W_E of Whitney squares satisfying conditions of Theorem 4.1. We also fix a smooth partition of unity subordinated to the Whitney decomposition W_E , see Lemma 7.2. There are various algorithms for constructing of these two classical elements of Whitney extension methods, see, e.g. Stein [43].

Step 2: Lacunae of Whitney squares, a graph of lacunae, and interior bridges. At this step of the pre-work we construct the family \mathcal{L}_E of lacunae of the set E , see Subsection 4.1. For every lacuna $L \in \mathcal{L}_E$ we fix all its contacting lacunae $L' \in \mathcal{L}_E$, $L' \leftrightarrow L$. See Subsection 5.2, Definition 5.5.

Then we construct an interior bridge (A_L, B_L) of the lacuna L . See Definition 5.8.

Step 3: Bridges between lacunae. Using the technique described in Section 6 we construct the family \mathcal{BR}_E of all bridges between lacunae. Simultaneously with constructing of bridges we determine pairs of *connected bridges* $T, T' \in \mathcal{BR}_E$, $T \longleftrightarrow T'$, as it was done in Section 6 for bridges satisfying conditions (6.1) and (6.7).

Then, basing on the algorithm suggested in Proposition 6.6, we construct a one-to-one mapping which to every pair (T, T') of connected bridges assigns a square $K(T, T')$ satisfying conditions (6.21) and (6.22) of the proposition.

As a result we obtain a one-to-one mapping defined on a family of well-separated squares \mathcal{K}_E which to every square $K \in \mathcal{K}_E$ assigns a pair of connected bridges $T_K, T'_K \in \mathcal{BR}_E$, $T_K \longleftrightarrow T'_K$. The ends of these bridges form a triangle $\Delta(K)$ (true or degenerate) as it

described in Subsection 8.2.

Step 4: The Menger curvature measure μ_E and “important” squares. We fix the set $\mathcal{C}_E = \{c_Q : Q \in \mathcal{K}_E\}$ of centers of the squares from the family \mathcal{K}_E , see (8.12). Then we construct the measure μ_E by formulas (8.13) and (8.14).

Basing on the approach suggested in Subsection 9.1, we determine the family of “important” squares generated by the measure μ_E . We do this in two steps. First for each square $K \in \mathcal{K}_E$ we construct the square \tilde{K} , see (9.6), whose “radius” $R_{\tilde{K}}$ satisfies inequalities (9.3) and (9.4). (Recall that $\tilde{K} \supset K$.)

We obtain a new family of squares $\tilde{\mathcal{K}}_E = \{\tilde{K} : K \in \mathcal{K}_E\}$. Then we extract from this family a subfamily \mathcal{Q}_E of “important” squares satisfying conditions (i) and (ii) of Proposition 9.1. We do this basing on a constructive filtering procedure suggested in the proof of this proposition.

Step 5: Centers of “important” squares and lacunary Whitney-type extensions. We fix the set $\mathcal{C} = \{c_Q : Q \in \mathcal{Q}_E\}$ of centers of the “important” squares. At this step of the pre-work we construct all necessary ingredients of a *lacunary Whitney-type extension* from the set \mathcal{C} . We have described this modification of the Whitney method in Remark 9.3.

First we fix a Whitney covering $W_{\mathcal{C}}$ of the open set $\mathbf{R}^2 \setminus \mathcal{C}$ and a smooth partition of unity $\{\psi_Q : Q \in W_{\mathcal{C}}\}$ subordinated to $W_{\mathcal{C}}$.

Then we determine a family $\mathcal{L}_{\mathcal{C}}$ of all lacunae of the set \mathcal{C} . Finally, we construct a “projection” of $\mathcal{L}_{\mathcal{C}}$ into \mathcal{C} , i.e., an “almost” one-to-one mapping $\mathcal{L}_{\mathcal{C}} \ni L \mapsto \mathcal{PR}(L) \in \mathcal{C}$ satisfying conditions (9.8) and (9.9).

All preparations are finished, and we turn to the second part of the algorithm.

10.2. Main Algorithm: from values of a function to its almost optimal $L_p^2(\mathbf{R}^2)$ -extension.

In this subsection we present main steps of the extension algorithm which to an arbitrary function $f : E \rightarrow \mathbf{R}$ assigns its almost optimal extension $F \in L_p^2(\mathbf{R}^2)$, $p > 2$. Furthermore, the extension F depends linearly on the function f . This algorithm uses only the values of the function f on E and the geometrical objects which we have constructed at the stage of the pre-work of the algorithm.

Step 1: the mapping $\mathcal{T}(f)$ and its averages on “important” squares. Using formulas (8.15) and (8.16), we construct the mapping $\mathcal{T}(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$; thus

$$\mathcal{T}(c_K; f) = \nabla P_{\Delta(K)}[f]$$

whenever $K \in \mathcal{K}_E$ and $\Delta(K)$ is a true triangle, and $\mathcal{T}(c_K; f) = 0$ if $\Delta(K)$ is a degenerate triangle. The mapping $\mathcal{T}(x; f) = 0$ for every point $x \in \mathbf{R}^2 \setminus \mathcal{C}_E$ (i.e., out of the family of centers of the squares from \mathcal{K}_E).

Recall that $P_{\Delta(K)}[f]$ is the affine polynomial which interpolates f on the vertices of $\Delta(K)$. Clearly, its gradient depends linearly on f so that the mapping $\mathcal{T}(f)$ depends linearly on f as well.

Let $\mathcal{C} = \{c_Q : Q \in \mathcal{Q}_E\}$ be the family of centers of the “important” squares, see Step 4 of the pre-work. We define a new mapping $\tilde{\mathcal{T}}_1(f) : \mathcal{C} \rightarrow \mathbf{R}^2$ such that for every “important”

square $Q \in \mathcal{Q}_E$

$$\tilde{\mathcal{T}}_1(c_Q; f) := \frac{1}{\mu_E(Q)} \int_Q \mathcal{T}(z; f) d\mu_E(z).$$

Note that, by definition of $\mathcal{T}(f)$ and the Menger curvature measure, see (8.14), for each $Q \in \mathcal{Q}_E$ we have

$$\mu_E(Q) = \sum \left\{ \mathbf{c}_{\Delta(K)}^p |K| : K \in \mathcal{K}_E, c_K \in Q \right\}$$

and

$$\tilde{\mathcal{T}}_1(c_Q; f) = \frac{1}{\mu_E(Q)} \sum \left\{ \nabla P_{\Delta(K)}[f] \mathbf{c}_{\Delta(K)}^p |K| : K \in \mathcal{K}_E, c_K \in Q \right\}.$$

Since the affine polynomials $P_{\Delta(K)}[f]$ depend linearly on f , the mapping $\tilde{\mathcal{T}}_1(f)$ depends linearly on f as well.

Step 2: the component $\mathcal{T}_1(f)$. At this step we extend the mapping $\tilde{\mathcal{T}}_1(f)$ from the set \mathcal{C} to all of \mathbf{R}^2 using the lacunary modification of the Whitney method described at Step 5 of the pre-work. More specifically, let $W_{\mathcal{C}}$ be the Whitney covering of $\mathbf{R}^2 \setminus \mathcal{C}$ introduced at this step. Let $L \in \mathcal{L}_{\mathcal{C}}$ be a lacuna of Whitney squares from $W_{\mathcal{C}}$, and let $Q \in L$. Recall that at Step 5 of the pre-work we have also constructed a “projection” $\mathcal{L}_{\mathcal{C}} \ni L \mapsto \mathcal{PR}(L) \in \mathcal{C}$. We put

$$A_Q := \mathcal{PR}(L) \quad \text{for every } Q \in L.$$

Then we construct a Whitney-type extension $\mathcal{T}_1(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ of the mapping $\tilde{\mathcal{T}}_1(f)$:

$$\mathcal{T}_1(x; f) := \begin{cases} \tilde{\mathcal{T}}_1(x; f), & x \in \mathcal{C}, \\ \sum_{Q \in W_{\mathcal{C}}} \psi_Q(x) \tilde{\mathcal{T}}_1(A_Q; f), & x \in \mathbf{R}^2 \setminus \mathcal{C}. \end{cases}$$

Here $\{\psi_Q : Q \in W_{\mathcal{C}}\}$ is a smooth partition of unity subordinated to $W_{\mathcal{C}}$ which we have determined at Step 5 of the pre-work.

Obviously, since $\tilde{\mathcal{T}}_1(f)$ depends linearly on f and the Whitney extension operator is a linear operator, the mapping $\mathcal{T}_1(f)$ depends linearly on f .

Step 3: pre-selections and selections. Let $L \in \mathcal{L}_E$ be an arbitrary lacuna of the set E , and let $T = T(L) = (A_L, B_L)$ be its interior bridge with the ends at points $A_L, B_L \in E$, see Step 2 of the pre-work. Using the formulas (8.21) and (8.22) at this step we construct a pre-selection $\tilde{g}(T; f)$ and a selection $g(T; f)$ of the set valued mapping G_f , see (8.1).

Thus we put

$$\tilde{g}(T; f) = \mathcal{T}_1(A_L; f),$$

and

$$g(T; f) = \text{Pr}(\tilde{g}(T; f); G_f(T)).$$

Recall that given a straight line $\ell \subset \mathbf{R}^2$ by $\text{Pr}(x; \ell)$ we denote the orthogonal projection of a point $x \in \mathbf{R}^2$ onto ℓ .

Clearly, since $\mathcal{T}_1(f)$ depends linearly on f , the same is true for the pre-selection $\tilde{g}(f)$.

Let us see that the selection $g(f)$ depends linearly on f as well. In fact, let us present the straight line

$$G_f(T) = \{z \in \mathbf{R}^2 : \langle z, A_L - B_L \rangle = f(A_L) - f(B_L)\}$$

in the form

$$G_f(T) = \{z \in \mathbf{R}^2 : \langle z, n_T \rangle = D_T(f)\}$$

where

$$n_T = \frac{A_L - B_L}{\|A_L - B_L\|_2} \quad \text{and} \quad D_T(f) = \frac{f(A_L) - f(B_L)}{\|A_L - B_L\|_2}.$$

By H_T we denote a one dimensional linear subspace of \mathbf{R}^2

$$H_T = \{z \in \mathbf{R}^2 : \langle z, n_T \rangle = 0\}.$$

Then

$$G_f(T) = H_T + D_T(f) n_T$$

so that for every $x \in \mathbf{R}^2$

$$\Pr(x; G_f(T)) = \Pr(x; H_T) + D_T(f) n_T.$$

In particular,

$$g(T; f) = \Pr(\tilde{g}(T; f); G_f(T)) = \Pr(\tilde{g}(T; f); H_T) + D_T(f) n_T.$$

Note that $\Pr(\cdot; H_T)$ is a linear operator, the vector function $f \rightarrow D_T(f) n_T$ depends linearly on f , and the same is true for the mapping $f \rightarrow \tilde{g}(f)$. Hence we conclude that the selection $g(f)$ depends linearly on f .

Step 4: “almost optimal” affine polynomials. At this step given an arbitrary lacuna $L \in \mathcal{L}_E$ and its interior bridge $T(L)$ with the ends at points $A_L, B_L \in E$ we construct an affine polynomial $P_L(f) \in \mathcal{P}_1$ such that

$$P_L(A_L; f) = f(A_L), \quad P_L(B_L; f) = f(B_L)$$

and

$$\nabla P_L(f) = g(T(L); f)$$

where $T(L) = (A_L, B_L) \in \mathcal{BR}_E$ is the interior bridge of the lacuna L . Thus

$$(10.1) \quad P_L(x; f) = f(A_L) + \langle g(T(L); f), x - A_L \rangle, \quad x \in \mathbf{R}^2.$$

As we have proved at the previous step, the vector function $f \rightarrow g(T(L); f)$ is a linear function of f , so that, by formula (10.1), the affine polynomial $P_L(f)$ depends linearly on f .

Step 5: the extension operator. This is the final step of the algorithm. We apply the lacunary Whitney-type extension operator suggested in Section 7 to the family of affine polynomials $\{P_L(f) : L \in \mathcal{L}_E\}$ from the previous step and construct the required extension of the function f .

Let $L \in \mathcal{L}_E$ be a lacuna. We put

$$P^{(Q)}(f) = P_L(f) \quad \text{for every square } Q \in L$$

where $P_L(f)$ is the polynomial defined by (10.1).

Finally, we construct the extension $F(f) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by the formula

$$F(x; f) := \begin{cases} f(x), & x \in E, \\ \sum_{Q \in W_E} \varphi_Q(x) P^{(Q)}(x; f), & x \in \mathbf{R}^2 \setminus E. \end{cases}$$

Here W_E and $\{\varphi_Q : Q \in W_E\}$ are the Whitney covering and the smooth partition of unity subordinated to W_E respectively. See Step 1 of the pre-work.

Since the polynomials of the family $\{P_L(f) : L \in \mathcal{L}_E\}$ depend linearly on f , *the extension operator $f \rightarrow F(f)$ is linear.*

11. Refinements of the trace criterion: Theorems 1.4, 1.6 and 1.11.

11.1. Proof of Theorem 1.4.

First we note that for *any* choice of objects in parts (i), (ii) and (iii) of Theorem 1.4, the quantity in the right-hand side of the theorem's equivalence does not exceed $C(p)\|f\|_{L_p^2(\mathbf{R}^2)|_E}$. This follows from the results of Subsection 3.1.

In Section 6 we construct a certain family of squares \mathcal{K}_E which plays an important role in our construction. In Subsection 8.2 we have assigned to \mathcal{K}_E a one-to-one mapping Δ from \mathcal{K}_E into the family $\text{Triangle}(E)$ of all triangles with vertices in E . We know that for each $K \in \mathcal{K}_E$ we have $\text{diam } \Delta(K) \sim \text{diam } K$ and $\Delta(K) \subset \gamma K$.

Then we construct a measure μ_E on \mathbf{R}^2 with $\text{supp } \mu_E \subset \mathcal{C}_E$, see (8.13). Here \mathcal{C}_E is the set of centers of squares from \mathcal{K}_E , see (8.12). (Recall that we call μ_E the Menger curvature measure generated by E .)

Prove that

$$(11.1) \quad \#\mathcal{K}_E \leq C \#E.$$

In fact, each lacuna $L \in \mathcal{L}_E$ generates a family of squares from \mathcal{K}_E using only its *contacting* squares of L , i.e., the squares which have common points with the squares from other lacunae. In other words, each lacuna $L' \in \mathcal{L}_E$ such that $L' \leftrightarrow L$ generates (together with L) a square $K \in \mathcal{K}_E$.

But, by Proposition 5.6, the number of lacunae L' which contact with L is bounded by an absolute constant C . Thus

$$\#\mathcal{K}_E \leq C \#\mathcal{L}_E.$$

In turn, by Corollary 5.4, $\#\mathcal{L}_E \leq C\#E$ proving (11.1).

Let us note that the family \mathcal{K}_E may be partitioned in a natural way into two families of squares: the family

$$\mathcal{K}_E^{(tr)} := \{K \in \mathcal{K}_E : \Delta(K) \text{ is a true triangle}\}$$

and the family

$$\mathcal{K}_E^{(dg)} := \{K \in \mathcal{K}_E : \Delta(K) \text{ is a degenerate triangle}\}.$$

Let us consider the family $\mathcal{K}_E^{(dg)}$. (Of course it can be empty for some E .) For each $K \in \mathcal{K}_E^{(dg)}$ the vertices of the degenerate triangle $\Delta(K)$ are three collinear points in \mathbf{R}^2 . Let us denote them by $a_1(K), a_2(K), a_3(K)$. We may assume that $a_2(K) \in (a_1(K), a_3(K))$.

Let us enumerate the family $\mathcal{K}_E^{(dg)}$:

$$\mathcal{K}_E^{(dg)} = \{Q_1, Q_2, \dots, Q_m\}.$$

Thus $m \leq \#\mathcal{K}_E \leq C\#E$. We also put

$$z_1^{(i)} := a_1(Q_i), \quad z_2^{(i)} := a_2(Q_i), \quad z_3^{(i)} := a_3(Q_i).$$

Remark that, since $\Delta(K) \subset \gamma K$, we have

$$z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \in E \cap (\gamma Q_i).$$

Examining the method of proof suggested in Section 8, the reader can readily see that the squares $\{Q_1, Q_2, \dots, Q_m\}$ and the triples of collinear points $\{z_1^{(i)}, z_2^{(i)}, z_3^{(i)}\}$ are the required objects in part (i) of Theorem 1.4.

We turn to the family $\mathcal{K}_E^{(tr)}$. We construct a non-negative Borel measure μ_E on \mathbf{R}^2 using the formulas (8.13) and (8.14). Then we construct a “gradient” mapping $\mathcal{T}(f)$ using the formulas (8.15) and (8.16).

At this point we modify the proof of Theorem 1.2 by applying to the mapping $\mathcal{T}(f)$ and the measure μ_E the following refinement of Theorem 8.3. (Here we present a vector version of this result).

Theorem 11.1 ([40], Subsection 6.1.) *Let $2 < p < \infty$ and let μ be a non-trivial non-negative Borel measure on \mathbf{R}^2 . There exist absolute constants $\gamma = \gamma > 0$ and $N \in \mathbf{N}$, a family \mathcal{Q} consisting of pairwise disjoint squares and a family $\tilde{\mathcal{Q}}$ of squares in \mathbf{R}^2 with covering multiplicity $M(\tilde{\mathcal{Q}}) \leq N$, mappings*

$$\mathcal{Q} \ni Q \mapsto Q' \in \tilde{\mathcal{Q}} \quad \text{and} \quad \mathcal{Q} \ni Q \mapsto Q'' \in \tilde{\mathcal{Q}}$$

satisfying the condition

$$Q' \cup Q'' \subset \gamma Q \quad \text{for all} \quad Q \in \mathcal{Q},$$

such that for every mapping $\mathcal{V} \in \vec{L}_{p,loc}(\mathbf{R}^2; \mu)$ from \mathbf{R}^2 into \mathbf{R}^2 its norm in the space $\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu)$ can be calculated (up to a constant depending only on p) as follows:

$$\|\mathcal{V}\|_{\vec{\Sigma}} \sim \left(\sum_{Q \in \mathcal{Q}} \frac{(\text{diam } Q)^{n-p} \iint_{Q' \times Q''} \|\mathcal{V}(x) - \mathcal{V}(y)\|^p d\mu(x) d\mu(y)}{\{(\text{diam } Q')^{n-p} + \mu(Q')\} \{(\text{diam } Q'')^{n-p} + \mu(Q'')\}} \right)^{\frac{1}{p}}.$$

The families of squares \mathcal{Q} and $\tilde{\mathcal{Q}}$ and the mappings $Q \mapsto Q'$ and $Q \mapsto Q''$ from the above theorem provides the objects from part (iii) of Theorem 1.4.

In turn, the objects of part (ii) of this theorem are the family of squares $\mathcal{K}_E^{(tr)}$ and the mapping $K \mapsto \Delta(K)$ defined on this family.

The method of proof suggested in Sections 7 and 8 and Theorem 11.1 shows that in all our considerations in these sections we may restrict ourself only to these particular objects (i.e., families \mathcal{Q} and \mathcal{K} , mappings $Q \mapsto Q'$ and $Q \mapsto Q''$, etc., determined in Theorem 11.1).

This enables us to put in the proof of Theorem 7.1 the number λ to be equal the right-hand side in the equivalence of Theorem 1.4. Then, following the method of proof given in Sections 7 and 8, we obtain the required inequality

$$\|f\|_{L_p^2(\mathbf{R}^2)}|_E \leq C \lambda^{\frac{1}{p}}.$$

This completes the proof of Theorem 1.4. ■

11.2. Proof of Theorem 1.6: sparsification.

The proof follows the same scheme as the proof of Theorem 1.4. The only difference that instead of Theorem 11.1 in this case we use the following

Theorem 11.2 ([40], Subsection 6.3) *Let μ be a non-trivial non-negative Borel measure on \mathbf{R}^2 , $2 < p < \infty$, and let*

$$\vec{\Sigma} = \vec{L}_p^1(\mathbf{R}^2) + \vec{L}_p(\mathbf{R}^2; \mu).$$

There exist families of closed sets $\{G_1, G_2, \dots\}$ and $\{H_1, H_2, \dots\}$ in \mathbf{R}^2 with covering multiplicity $M(\{G_i\}), M(\{H_i\}) \leq C$ where C is an absolute constant, and a family $\{\lambda_1, \lambda_2, \dots\}$ of positive numbers such that for every mapping $\mathcal{V} \in \vec{L}_{p,loc}(\mathbf{R}^2; \mu)$ from \mathbf{R}^2 into \mathbf{R}^2 the following equivalence

$$\|\mathcal{V}\|_{\vec{\Sigma}}^p \sim \sum_{i=1}^{\infty} \lambda_i \iint_{G_i \times H_i} |\mathcal{V}(x) - \mathcal{V}(y)|^p d\mu(x) d\mu(y)$$

holds. The constants of this equivalence depend only on p .

Combining this results with ideas suggested in the proof of Theorem 1.4, we obtain the following trace criterion:

Theorem 11.3 *Let $2 < p < \infty$ and let E be a finite subset of \mathbf{R}^2 . There exist absolute constants $C > 0$ and $N \in \mathbf{N}$ and:*

(i) *A family $\{Q_i : i = 1, \dots, \ell\}$, $\ell \leq C \#E$, of pairwise disjoint squares and a family*

$$\{z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \in E : z_2^{(i)} \in (z_1^{(i)}, z_3^{(i)}), i = 1, \dots, \ell\}$$

of triples of collinear points;

(ii) *A family \mathcal{K} of pairwise disjoint squares with $\#\mathcal{K} \leq C \#E$, and a mapping*

$$\mathcal{K} \ni K \mapsto \Delta(K) \in \text{Triangle}(E);$$

(iii) *Families of closed sets*

$$\mathcal{U} = \{U_1, U_2, \dots, U_m\} \quad \text{and} \quad \mathcal{V} = \{V_1, V_2, \dots, V_m\}, \quad m \leq C \#E,$$

in \mathbf{R}^2 with covering multiplicity $M(\mathcal{U}) + M(\mathcal{V}) \leq N$, and a family of positive numbers $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$,

such that for every function $f : E \rightarrow \mathbf{R}$ the following equivalence

$$\begin{aligned} \|f\|_{L_p^2(\mathbf{R}^2)|_E}^p &\sim \sum_{i=1}^{\ell} \left| \frac{f(z_1^{(i)}) - f(z_2^{(i)})}{\|z_1^{(i)} - z_2^{(i)}\|_2} - \frac{f(z_2^{(i)}) - f(z_3^{(i)})}{\|z_2^{(i)} - z_3^{(i)}\|_2} \right|^p (\text{diam } Q_i)^{2-p} \\ &+ \sum_{i=1}^m \alpha_i S_p(f : U_i, V_i; \mathcal{K}) \end{aligned}$$

holds. The constants of this equivalence depend only on p .

Recall that given sets $U, V \subset \mathbf{R}^2$

$$S_p(f : U, V; \mathcal{K}) := \sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in U}} \sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in V}} \|\nabla P_{\Delta(K')}[f] - \nabla P_{\Delta(K'')}[f]\|^p \mathbf{c}_{\Delta(K')}^p |K'| \mathbf{c}_{\Delta(K'')}^p |K''|.$$

We are in a position to finish the proof of Theorem 1.6.

First we note that the first sum in the above equivalence consists of at most $C \#E$ linear functionals to the power p each depending on 3 values of f on E .

Let us determine similar linear functionals for the second sum. (But in this case each of these functionals will depend on at most *six* points of E .)

We shall do this using the so-called *spectral sparsification* of the quantity $S_p(f : U_i, V_i; \mathcal{K})$, $i = 1, 2, \dots, m$. We will be needed a certain special form of matrix sparsification related to the existence of p -sparsifiers of non-negative product matrices.

Theorem 11.4 (*B. Klartag*) *Let $2 < p < \infty$. Let $G = (g_i g_j)$ be $n \times n$ matrix with $g_i \geq 0, i = 1, \dots, n$. Then there exists an $n \times n$ matrix $H = (h_{ij})$, $h_{ij} \geq 0$, such that:*

- (i) $\text{supp}(H) \subseteq \text{supp}(G)$;
- (ii) $\#(\text{supp}(H)) \leq C n$;
- (iii) *For every $x \in \mathbf{R}^n$ we have*

$$\sum_{i=1}^n \sum_{j=1}^n g_i g_j |x_i - x_j|^p \leq \sum_{i=1}^n \sum_{j=1}^n h_{ij} |x_i - x_j|^p \leq C(p) \sum_{i=1}^n \sum_{j=1}^n |x_i - x_j|^p.$$

We refer to H as a p -sparsifier of G .

We recall that the remarkable sparsification theorem of J. D. Batson, D. A. Spielman and N. Srivastava [1] states that for every (not necessarily product) non-negative matrix $G = (g_i g_j)$ there exists a “good” 2-sparsifier.

The proof of Theorem 11.4 follows from the next statement proven by B. Klartag: *if $p \in (2, \infty)$ and H is a 2-sparsifier of a non-negative product matrix G (with a certain constant in (iii)), then H is a p -sparsifier of G (with a bigger constant depending on p).* A. Naor [30] showed that if G is not a product non-negative matrix, then in general it is not true.

We note that the work [1] have been remarked by C. Fefferman, A. Israel and G. K. Luli [20] as a useful tool in solution to Problem 1.5 and related problems.

Let us apply Theorem 11.4 to the quantity $S_p(f : U, V; \mathcal{K})$ whenever U and V are two arbitrary subsets of \mathbf{R}^2 . We obtain the following

Theorem 11.5 *Let $2 < p < \infty$. There exists a function $h = h(K', K'') \geq 0$ defined on the set*

$$\{K' \in \mathcal{K} : c_{K'} \in U\} \times \{K'' \in \mathcal{K} : c_{K''} \in V\}$$

which satisfies the following conditions:

- (i) $h(K', K'') \neq 0$ at most for $C\#(U \cup V)$ pairs (K', K'') ;
- (ii) *The following equivalence*

$$S_p(f : U, V; \mathcal{K}) \sim H_p(f : U, V; \mathcal{K})$$

holds with constants depending only on p . Here

$$H_p(f : U, V; \mathcal{K}) := \sum_{\substack{K' \in \mathcal{K} \\ c_{K'} \in U}} \sum_{\substack{K'' \in \mathcal{K} \\ c_{K''} \in V}} h(K', K'') \|\nabla P_{\Delta(K')}[f] - \nabla P_{\Delta(K'')}[f]\|^p.$$

Using this theorem we finish the proof of Theorem 1.6 as follows.

Given a vector $x = (x_1, x_2) \in \mathbf{R}^2$ we put $(x)_1 := x_1$ and $(x)_2 := x_2$. Then for each $K', K'' \in \mathcal{K}$ we have the following equivalence

$$\|\nabla P_{\Delta(K')}[f] - \nabla P_{\Delta(K'')}[f]\|^p \sim |\lambda_1(f; K', K'')|^p + |\lambda_2(f; K', K'')|^p$$

where

$$\lambda_i(f; K', K'') := (\nabla P_{\Delta(K')}[f] - \nabla P_{\Delta(K'')}[f])_j, \quad j = 1, 2.$$

Clearly, $\lambda_1(f; K', K'')$ and $\lambda_2(f; K', K'')$ are linear functional each depending on six values of f on E .

Let

$$Y_i := \{c_K : K \in \mathcal{K}, c_K \in U_i\}, \quad i = 1, \dots, m,$$

and

$$X_i := \{c_K : K \in \mathcal{K}, c_K \in V_i\}, \quad i = 1, \dots, m,$$

and let $\mathcal{Y} = \{Y_i\}$ and $\mathcal{X} = \{X_i\}$. Since the families $\mathcal{U} = \{U_i\}$ and $\mathcal{V} = \{V_i\}$ has covering multiplicity $M(\mathcal{U}) + M(\mathcal{V}) \leq N$, we conclude that

$$(11.2) \quad M(\mathcal{Y}) + M(\mathcal{X}) \leq N$$

as well.

Thus, by Theorem 11.5 each quantity $S_p(f : U_i, V_i; \mathcal{K})$ can be represented as a sum of the p -powers of at most $C(\#Y_i + \#X_i)$ linear functionals, so that the second sum in the equivalence of Theorem 11.3 depends on at most

$$L \leq C \sum_{i=1}^m (\#Y_i + \#X_i)$$

linear functionals. But, by (11.2),

$$\sum_{i=1}^m (\#Y_i + \#X_i) \leq N \# \left\{ \bigcup_{i=1}^m (Y_i \cup X_i) \right\} \leq N \# \mathcal{K}.$$

It remains to note that, by part (ii) of Theorem 1.4, $\# \mathcal{K} \leq C \# E$ so that the number L of the linear functionals is bounded by

$$L \leq C \# E.$$

Theorem 1.6 is completely proved. ■

11.3. A sketch of the proof of Theorem 1.11. We note that, by definitions (1.7) and (1.7), given a disk $B \subset \mathbf{R}^2$ the condition $\mathbf{c}_p(B; \mathcal{D}, \Delta) \leq c_B$ is equivalent to the inequality

$$\sum_{c_D \in B} \mathbf{c}_{\Delta(D)}^p |D| \leq |B| / r_B^p = \pi r_B^{2-p}.$$

We let $\tilde{\sigma}_p(B; \mathcal{D})$ denote the quantity

$$\tilde{\sigma}_p(B; \mathcal{D}) := \sum_{c_D \in B} \mathbf{c}_{\Delta(D)}^p |D|.$$

This quantity is an analog of the quantity $\sigma_p(Q; \mathcal{K})$ defined by (1.5). Thus the condition (1.9) of the theorem is equivalent to the inequality

$$(\text{diam } B')^{p-2} \tilde{\sigma}_p(B'; \mathcal{D}) + (\text{diam } B'')^{p-2} \tilde{\sigma}_p(B''; \mathcal{D}) \leq C(p).$$

Obviously, this inequality is a “disk” version of inequality (7.2).

We note that an analogue of Theorem 1.11 for squares rather than for disks is also true. Its necessity part follows from the necessity part of Theorem 1.2. The sufficiency part of such an analogue follows from Theorem 7.1.

We also note that we can slightly modify the condition (7.2) in formulation of Theorem 7.1. More specifically, we can replace this condition by a more general one:

$$(11.3) \quad (\text{diam } Q')^{p-2} \sigma_p(Q'; \mathcal{K}) + (\text{diam } Q'')^{p-2} \sigma_p(Q''; \mathcal{K}) \leq \eta$$

where η is a positive constant. The result of Theorem 7.1 remains true after such a modification, but with constant C in inequality (7.1) depending on p and also on η . In fact, in the proof of this theorem we use inequality (11.3) only to verify condition (8.17) in Theorem 8.3. But obviously this condition holds with $\tau = \eta$.

The proof of Theorem 1.11 follows the same scheme: we repeat our considerations for families of disks rather than squares. There are no any technical difficulties in such a generalization of the methods and ideas developed for squares to the case of disks. We will only remark two places in the proof where certain non-trivial changes should be done.

First of them relates to an analogue of the Whitney covering Theorem 4.1 for disks. Of course, in this case we can not cover the open set $\mathbf{R}^2 \setminus E$ by *non-overlapping* disks D such that $\text{diam } D \sim \text{dist}(D, E)$. Nevertheless for our purpose it suffice to cover $\mathbf{R}^2 \setminus E$ by a

family \widetilde{W}_E of disks whose *covering multiplicity* is bounded by an absolute constant N . In other words, every point $x \in \mathbf{R}^2$ is covered at most N disks from the family \widetilde{W}_E .

The existence of a Whitney-type covering of such a kind, i.e., a covering of an open set by a family of Whitney disks with finite multiplicity, follows from a general result proven by M. Guzman [23]. (In turn, this result is based on the Besicovitch covering theorem [2].)

Our second remark relates to Theorem 8.3 which is an important ingredient of the proof of Theorem 7.1. For its analogue for disks (rather than squares) we refer the reader to the paper [40]. (See there Remark 6.7 and Theorem 6.8.) \blacksquare

Remark 11.6 Analyzing the proof of the sufficiency we note that either the disk B' or the disk B'' from part (ii) of Theorem 1.11 is an “*important*” disk, i.e., either equivalence $\mathbf{c}_p(B' : \mathcal{D}, \Delta) \sim \mathbf{c}_{B'}$ or equivalence $\mathbf{c}_p(B'' : \mathcal{D}, \Delta) \sim \mathbf{c}_{B''}$ holds with absolute constants. This enables us to replace the condition (1.9) by a slightly stronger condition

$$\alpha \leq \mathbf{c}_p(B' : \mathcal{D}, \Delta) / \mathbf{c}_{B'} + \mathbf{c}_p(B'' : \mathcal{D}, \Delta) / \mathbf{c}_{B''} \leq 1$$

where α is an absolute positive constant. \triangleleft

Remark 11.7 In part (ii) of Theorem 1.2 we require that the squares Q' and Q'' belong to the family \mathcal{Q} . Note that the theorem’s result remains true if we replace this condition with the following one: $Q', Q'' \in \widetilde{\mathcal{Q}}$ where $\widetilde{\mathcal{Q}}$ is an arbitrary family of pairwise disjoint squares in \mathbf{R}^2 .

The necessity of this modification of part (ii) of Theorem 1.2 directly follows from the method of proof suggested in Section 3. In particular, in formulation of Theorem 3.7 the families \mathcal{A} and \mathcal{S} may be different. This enables us in the proof of Proposition 3.8 to replace (3.19) with the following definition:

$$\mathcal{A} = \mathcal{Q}, \quad \mathcal{S} = \widetilde{\mathcal{Q}}.$$

In turn, the sufficiency follows from the proof of the sufficiency part of Theorem 1.2 where we can put $\widetilde{\mathcal{Q}} = \mathcal{Q}$. \triangleleft

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